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Instanton corrections to circular Wilson loops in $\mathcal{N}=4$ Supersymmetric Yang-Mills

Massimo Bianchi
Dipartimento di Fisica, Università di Roma "Tor Vergata"
Via della Ricerca Scientifica, 1, 00133 Roma, Italy
E-mail: Massimo.Bianchi@roma2.infn.it

Michael B. Green and Stefano Kovacs
Department of Applied Mathematics and Theoretical Physics
Wilberforce Road, Cambridge CB3 0WA, UK
E-mail: M.B.Green@damtp.cam.ac.uk, S.Kovacs@damtp.cam.ac.uk

Abstract: It is argued that whereas supersymmetry requires the instanton contribution to the expectation value of a straight Wilson line in the $\mathcal{N} = 4$ supersymmetric SU(2) Yang-Mills theory to vanish, it is not required to vanish in the case of a circular Wilson loop. A non-vanishing value can arise from a subtle interplay between a divergent integral over bosonic moduli and a vanishing integral over fermionic moduli. The one-instanton contribution to such Wilson loops is explicitly evaluated in semi-classical approximation. The method utilizes the symmetries of the problem to perform the integration over the bosonic and fermionic collective coordinates of the instanton. The integral is singular for small instantons touching the loop and is regularized by introducing a cutoff at the boundary of the (euclidean) AdS_5 moduli space. In the case of a circular loop a non-zero finite result is obtained when the cutoff is removed and a perimeter divergence subtracted. This is contrasted with the case of the straight line where the result is zero after subtraction of an identical divergence per unit length. The linear divergence is an artifact of our non-supersymmetric regulator that deserves further consideration. The generalization to gauge group SU(N) with arbitrary N is straightforward in the limit of small 't Hooft coupling. The extension to strong 't Hooft coupling is more challenging and only a qualitative discussion is given of the AdS/CFT correspondence.

Keywords: Non-perturbative Effects, Solitons Monopoles and Instantons, AdS/CFT and dS/CFT Correspondence, Extended Supersymmetry.

Contents

1. Introduction

The correspondence between $\mathcal{N}=4$ supersymmetric SU(N) Yang-Mills theory and type-IIB superstring theory on $AdS_5 \times S^5$ has been the subject of extensive study and is by now well tested in the large-N limit (see for instance [1]–[4]). Many of these tests involve comparison of correlation functions of gauge-invariant Yang-Mills operators with corresponding amplitudes in type-IIB supergravity (the small α'/ℓ^2 limit of superstring theory in $AdS_5 \times S^5$, where ℓ^2 is the scale of the curvature). By comparison, the correspondence involving non-local Yang-Mills operators, such as Wilson loops, has been relatively little studied. This is one of the main motivations for this paper.

The Wilson loop in pure Yang-Mills theory is the expectation value of the holonomy, $\langle W(\mathcal{C})\rangle = \langle \mathrm{Tr}\, \mathcal{P} \exp i \int_{\mathcal{C}} A_\mu dx^\mu \rangle$, which is a functional of an arbitrary curve $\mathcal{C}$. It is of central importance since it is an order parameter that characterizes the different phases of the theory. When the trace is taken in the fundamental representation of the gauge group G, such as the N-dimensional representation of SU(N), the Wilson loop decreases as the exponential of the area in a confining phase but as the exponential of the perimeter in a non-confining phase. A long rectangular loop determines the static potential between elementary test charges, such as quark charges in QCD. Similar considerations apply to Wilson loops in more general gauge theories in which there are additional dynamical fields in the adjoint representation. In supersymmetric theories the concept of the gauge connection generalizes to a superfield that contains other components beyond the usual vector potential. Correspondingly, the Wilson loop generalizes in a natural manner to include contributions from the extra fields. The $\mathcal{N}=4$ supersymmetric Yang-Mills theory is worthy of study in its own right since it is an example of a non-trivial superconformal field theory in four dimensions and a prototype for more general and realistic gauge theories. The natural generalization of the Wilson loop in this theory is defined by

$$\langle W(\mathcal{C})\rangle = \frac{1}{N}\left\langle \mathrm{Tr}\, \mathcal{P} \exp\left\{ i \int_{\mathcal{C}} \left(A_\mu \dot{x}^\mu + i\varphi_i \dot{y}^i + [\bar{\theta}_A \dot{x}_\mu \sigma^\mu \lambda^A + \theta^A \dot{y}_i \widehat{\Gamma}^i_{AB} \lambda^B + \mathrm{h.c.}] + \cdots \right) \times \right.\right.$$
$$\left.\left. \times\, ds \right\}\right\rangle , \tag{1.1}$$

where λ^A ($A=1,2,3,4$ indicates a $\mathbf{4}$ of the R-symmetry group, SU(4)) and φ_i ($i=1,\ldots,6$ labels the $\mathbf{6}$ of SU(4)) are the fermion and scalar fields in the $\mathcal{N}=4$ supermultiplet. The matrices $\widehat{\Gamma}^i_{AB}$ and $\bar{\widehat{\Gamma}}^{i\,AB}$ are SO(6) Clebsch-Gordan coefficients that couple a $\mathbf{6}$ to two $\mathbf{4}$'s and to two $\bar{\mathbf{4}}$'s, respectively. The curve $\mathcal{C}$ now represents the curve in "superspace" — in other words this kind of Wilson loop depends not only on the curve $x^\mu(s)$ but also on six extra variables $y^i(s)$ and on the spinors $\theta^A(s)$ and $\bar{\theta}_A(s)$ that contain the sixteen odd (Grassmann) variables of $\mathcal{N}=4$ on-shell superspace. The $\cdots$ in (1.1) stands for terms involving higher powers of the fermionic coordinates, θ^A and $\bar{\theta}_A$. The expression (1.1) is appropriate to euclidean signature whereas the factor of i in the coefficient of φ_i is absent with Minkowski signature. Its presence is important, among other reasons, because it implies that the exponential is not purely a phase. This expression can be motivated in various ways. For example, the loop can be considered to be the holonomy of an infinitely massive W-boson that is generated by breaking the gauge group $\mathrm{SU}(N+1)$ to $\mathrm{SU}(N)\times\mathrm{U}(1)$ (as shown in an appendix of [5]).

Wilson loops of this kind were first studied in [6, 7], where their interpretation within the AdS/CFT correspondence was stressed. Within string theory the Wilson loop is interpreted as the functional integral over all world-sheets embedded in AdS_5 and bounded by the loop. In the supergravity limit (the small α' limit of string theory) this integration over fluctuating surfaces is dominated by the surface of minimum area $A_{\rm min}$ in AdS_5. The behaviour of the loop is therefore $\langle W(\mathcal{C})\rangle \sim \exp(-A_{\rm min})$. Since the metric is singular near the boundary of AdS_5 an infinite perimeter term arises, representing the mass of the test particle circulating in the loop (it was shown in [5] that this divergence may be eliminated by an appropriate choice of world-sheet boundary conditions). The finite minimal area obtained by subtracting the divergent piece is a unique and well defined quantity [8].

As shown in [9] the expression (1.1) is invariant under κ transformations of the one-dimensional theory on the test particle world-line, provided the curve satisfies appropriate conditions and the Yang-Mills fields satisfy their equations of motion. This is closely related to the κ symmetry of the massless ($p^2 = 0$) ten-dimensional superparticle. A standard argument based on gauge-fixing of κ symmetry then implies that the loop is invariant under half of the 32 superconformal supersymmetries. In that case the supersymmetries are defined by spinor parameters κ^A_α, $\bar\kappa^{\dot\alpha}_A$ that are related by

$$\dot x_\mu\, \sigma^\mu\, \bar\kappa_A = \dot y_i\, \widehat\Gamma^i{}_{AB}\kappa^B\,. \tag{1.2}$$

It is easy to see that this constrains $\dot y^i$ so that

$$\dot y^i = n^i |\dot x|\,, \tag{1.3}$$

where n^i is an arbitrary fixed unit vector on the five-sphere ($n^2 = 1$). As we will see later the solutions to (1.2) have the form[1] $\kappa^A_\alpha = \eta^A_{\oplus\,\alpha} + (\sigma\cdot x\bar\xi^A_\oplus)_\alpha$, $\bar\kappa^{\dot\alpha}_A = \bar\eta^{\dot\alpha}_{\oplus\,A} + (\bar\sigma\cdot x\xi_{\oplus\,A})^{\dot\alpha}$, where the sixteen Poincaré supersymmetry parameters, $\eta^A_{\oplus\,\alpha}$ and $\bar\eta^{\dot\alpha}_{\oplus\,A}$, and the sixteen conformal supersymmetry parameters, $\bar\xi^A_{\oplus\,\dot\alpha}$ and $\xi^\alpha_{\oplus\,A}$, are related. We will only consider the special loops in which we set $\theta^A(s) = 0$ (so the terms in (1.1) that depend on θ are absent). A particularly symmetric example of a Wilson loop satisfying (1.2) is the circular loop of radius R. Superconformal invariance implies that the expectation value of such a loop cannot depend on R so that $\langle W\rangle_{\rm circle}$ is a constant, but it does depend in a non-trivial way on the dimensionless parameters $g_{\rm YM}$ and N. The special feature of a circular loop is that (1.2) implies that an *x-independent* linear combination of Poincaré and conformal supersymmetries remains unbroken. In other words, the sixteen unbroken supersymmetries are global whereas they are x-dependent for a generic loop satisfying (1.2). However, in the quantum theory it is necessary to introduce a cut off. As we will see later, this necessarily breaks the remaining supersymmetries and such Wilson loops receive quantum corrections. A class of perturbative contributions to the expectation values of Wilson loops of this kind has been calculated to all orders in the coupling constant and it has been argued that it contains all the relevant contributions, at least in the large-N limit [10]. This

[1] The subscript $_\oplus$ is used to label the parameters to avoid later confusion with the instanton moduli.

consists of the "rainbow diagrams" — the class of planar diagrams in which all propagators begin and end on the loop (so there are no internal interaction vertices). The sum of such diagrams was determined in terms of a zero-dimensional gaussian matrix model. A suggestion has been made [11] for extending this to all orders in the $1/N$ expansion (as well as all orders in the 't Hooft coupling) by use of an anomaly argument. This was arrived at by considering a "straight" Wilson line on $\mathbb{R}^4$. This case is even more special since (1.2) now implies that a subset of the Poincaré supersymmetries are unbroken and do not mix with the conformal supersymmetries. The unbroken Poincaré supersymmetries (but not the conformal supersymmetries) are preserved in the quantum theory in the presence of a suitable cutoff and protect the Wilson line expectation value so that $\langle W\rangle_{\rm line} = 1$. A circular loop passing through the origin is mapped to a straight line by a conformal inversion and it is the associated conformal anomaly that gives rise to a non-trivial expression for $\langle W\rangle_{\rm circle}$ as a function of the coupling. The argument in [11] suggested that the gaussian matrix model results should be taken seriously for all values of N, not simply in the large-N limit contemplated by [10]. However, as shown in [12], there is a wide class of matrix models with non-trivial potentials which give rise to the same leading N behaviour but give different expressions for the Wilson loop at finite N. Recent computations [13] have further questioned the validity of the conjecture within perturbation theory. From our perspective, it is notable that the expression for the circular Wilson loop suggested by [11] has no instanton contributions and does not depend on the vacuum angle, ϑ.

In this paper we will explicitly compute the one-instanton contribution to a circular Wilson loop in SU(2) $\mathcal{N}=4$ Yang-Mills in semi-classical approximation — to lowest order in the Yang-Mills coupling constant, $g_{\rm YM}$. A preliminary outline of this work was presented in [14].

1.1 Expectations based on supersymmetry

Before carrying out the calculations in detail it is of interest to use the symmetries of the problem to anticipate the result.[2] It is instructive to contrast the expression for the one-instanton contribution to the circular Wilson loop with that for the "straight line" (of the kind considered in [11]). Naively (ignoring the need for a cutoff), the loop preserves half of the Poincaré and conformal supersymmetries. At least some of these are broken by the presence of an instanton. For every supersymmetry of the background that is broken by the instanton there is a "true" fermionic modulus (a fermionic integration variable that does not enter into the integrand) so the integration over supermoduli space vanishes. However, this argument is too naive since the integration over the bosonic moduli, which parameterize AdS_5, diverges on the boundary, i.e. for instantons of small scale size. It is therefore essential to introduce a cutoff. In principle, such a cutoff can be introduced by considering the SU(N) theory as the limit of a SU($N+1$) theory spontaneously broken to SU(N) $\times$ U(1), in which the scalar field vacuum expectation value, M, becomes infinite [5]. The W-bosons have mass M and are in the N, $\bar{N}$ of SU(N). In the limit $M \to \infty$ the Wilson loop can be defined in terms of the holonomy of a W-boson with a specified trajectory. The Wilson loop can be regulated by keeping the mass, M, finite but large (compared to

[2] We are grateful to Juan Maldacena for conversations on the following points.

the inverse radius of the loop) in much the same way as considered in [6]. In this case the fluctuations of the test particle are non zero and the loop is smeared out over a region M^{-1}.

In the absence of the loop the cut-off theory preserves the sixteen Poincaré supersymmetries (with parameters $\eta_\oplus, \bar{\eta}_\oplus$) but breaks the sixteen conformal supersymmetries (with parameters $\xi_\oplus, \bar{\xi}_\oplus$), since the cut-off theory is not conformally invariant. We now want to consider the cut-off theory in the presence of the Wilson loop, which breaks further supersymmetries, as determined by the condition (1.2). It is easy to see from this equation that in the case of a straight line ($\dot{x}_\mu \propto x_\mu$) the Poincaré supersymmetries of opposite chiralities are related to each other ($\eta_\oplus$ is related to $\bar{\eta}_\oplus$). Therefore, there are eight residual supersymmetries of the Wilson line background in the cut-off theory. In the case of the circular loop the condition (1.2) relates the Poincaré supersymmetries to the conformal supersymmetries. Since the conformal supersymmetries are already broken by the cutoff we see that a circular loop in the cut-off theory does not preserve any supersymmetry. Later, we will also consider a cutoff that preserves a SO(5) symmetry instead of Poincaré symmetry. Such a cutoff is very natural when considering circular loops on S^4, which is convenient for displaying conformal symmetry. We will argue that a SO(5)-invariant cutoff cannot preserve any supersymmetry, even in the absence of the loop.

Now consider the introduction of the instanton. In the case of the straight line with the Poincaré-invariant cutoff the eight unbroken supersymmetries are broken by the instanton. This generates eight true fermionic moduli so the integrated expression for the instanton contribution should vanish. More generally, the presence of unbroken supersymmetries with this cutoff requires $\langle W\rangle_{\rm line} = 1$, as seen in the perturbative sector [10, 11] and in line with expectations based on the AdS/CFT correspondence [6]. In the case of the circular Wilson loop there are no surviving supersymmetries to be broken in the presence of the cutoff (whether it is Poincaré invariant or SO(5) invariant) so we conclude that the instanton contribution to the loop expectation value can be non zero and must be independent of the cutoff.

In the following we will make use of the standard BPST instanton solution of the Yang-Mills equations. In principle, these equations should be modified to include the effect of the Wilson loop source, which changes the standard BPST instanton solution of the sourceless theory. However, at least when the instanton is not too singular, the corrections to the equations induced by the current source are suppressed by powers of $g_{\rm YM}$ so they can be neglected in the semi-classical approximation. The singular configuration, in which a small instanton touches the loop, should be regulated in the Poincaré-invariant manner described above. In practice, we will regulate the singularity by introducing a cutoff in the integral over the collective coordinates of the instanton near the boundary of AdS_5. This cutoff breaks supersymmetry and potentially introduces an ambiguity in the finite value of the expectation value of the circular Wilson loop which will be discussed in the last section.

1.2 Strategy and layout

As usual, the instanton computation boils down to an integral over the supermoduli space spanned by eight bosonic and sixteen fermionic collective coordinates. The bosonic collective coordinates correspond to broken translation, scale and gauge symmetries and

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will be denoted by $(x_0^\mu, \rho_0, \alpha_0^a)$, respectively (although the gauge moduli will be irrelevant in the following). The fermionic collective coordinates are associated with broken Poincaré supersymmetries and conformal supersymmetries and will be denoted by $(\eta^A, \bar{\xi}^A)$, respectively. The prospect of directly evaluating even the bosonic part of the Wilson loop is somewhat daunting. However, considerable simplification arises after taking advantage of the conformal symmetries of this system. The presence of the loop breaks the SO(4, 2) conformal invariance but, as we will show in section 2, for a circular loop there is a residual unbroken $\mathrm{SO}(2,2) = \mathrm{SO}(2,1) \times \mathrm{SO}(2,1)$ subgroup. When considering the instanton calculation we will be interested in the euclidean version of the theory in which the AdS_5 boundary is S^4. The euclidean conformal group is SO(5, 1) and the subgroup that leaves the loop invariant is $\mathrm{SO}(3) \times \mathrm{SO}(2,1)$. We will further show that the presence of a circular loop in the $\mathcal{N} = 4$ supersymmetric theory breaks the full SU(2, 2|4) superconformal symmetry to a residual OSp(2, 2|4), which has sixteen fermionic generators.

These symmetries will be used to determine the structure of the instanton contribution to the Wilson loop in a toy bosonic model in section 3. This calculation includes only the bosonic moduli of the complete $\mathcal{N} = 4$ Wilson loop calculation (and is *not* the same as the expression for the instanton contribution to a Wilson loop in pure Yang-Mills theory, which has a non conformally-invariant measure.[3]) In order to streamline the discussion of the conformal properties it will prove convenient to make use of Dirac's formalism [15] for representing the conformal group by extending four-dimensional Minkowski space-time to six dimensions with signature (4, 2) with coordinates X_M $(M = 0, \ldots, 5)$, where X_0 and X_4 are time-like (and X_0 will be Wick rotated when describing the euclidean theory). In this way, the SO(4, 2) and SO(2, 2) symmetries can be represented linearly by rotations (and boosts) on X_M. Imposing the invariant constraint $X^M X_M = \ell^2$, where ℓ is an arbitrary dimensional constant, results in a representation of SO(4, 2) in AdS_5 and thence to a four-dimensional representation on the boundary of AdS_5. This will be reviewed in section 3.1 and appendix A.

The coordinates of euclidean AdS_5 (which is the ball, B^5) enter as collective coordinates in the instanton problem. The fact that the instanton is invariant in form under arbitrary conformal transformations, up to an irrelevant gauge transformation, together with the invariance of the loop under $\mathrm{SO}(3) \times \mathrm{SO}(2,1)$, will be used to evaluate the Wilson loop integrand. The expression for the integrand of the Wilson loop with an instanton at a generic point in moduli space, (x_0^μ, ρ_0), is identical to that in which the instanton is located at any point on the same $\mathrm{SO}(3) \times \mathrm{SO}(2,1)$ orbit. In particular, it is the same as if the instanton were at the centre of the loop, $\tilde{x}_0^\mu = 0$ with a scale $\tilde{\rho}(x_0, \rho_0)$ that is a certain $\mathrm{SO}(3) \times \mathrm{SO}(2,1)$-invariant function of X_M. But the expression for the Wilson loop with an instanton at the centre reduces to one with an abelian gauge field, in which case the path ordering is trivial which makes the integration over the moduli very simple. The integral is very divergent since there is no suppression of instantons located arbitrarily far from the loop. This type of divergence does not appear in the $\mathcal{N} = 4$ supersymmetric case. However, there is also a divergence from small scale instantons touching the loop,

[3] The complete calculation in the Yang-Mills theory, even to lowest non-trivial order in the coupling, is significantly more subtle.

which also needs to be addressed in the supersymmetric case. All of these divergences are regulated by imposing an SO(5)-invariant cutoff on the integration over the bosonic moduli that excludes a small spherical shell close to the boundary of the moduli space, AdS_5.

In section 3.3 we will consider the expression for the instanton contribution in the bosonic toy model to the straight Wilson line, which is defined on $\mathbb{R}^4$. This was the starting configuration considered in [11]. In the case of a straight line it is natural to use a Poincaré invariant cutoff, $\rho_0 \geq \epsilon$ (where ϵ is an infinitesimal constant), which is invariant under the translational isometry of the straight line. Both cutoffs (indeed, all possible cutoffs) break conformal invariance. The distinction between the SO(5)-invariant and Poincaré invariant cutoffs is crucial in determining the difference between the expectation values of the circular Wilson loop and straight Wilson line in the superconformal theory considered later.

In sections 4 and 5 we will describe the extension of these ideas to the $\mathcal{N}=4$ supersymmetric case in which there are sixteen fermionic moduli in addition to the bosonic moduli. The bosonic fields φ^i and A_μ have zero modes that are induced by the couplings to the fermionic sources in the usual manner. The multiplet of these zero modes can be generated from the classical instanton profile of the vector potential by successive application of the supersymmetries that are broken by the instanton. This leads to an expression for φ that is polynomial in fermionic moduli, beginning with a quadratic term

$$\hat{\varphi}^{ia} = \frac{1}{2} F^{a-}_{\mu\nu} \widehat{\Gamma}^i_{AB} \zeta^A \sigma^{\mu\nu} \zeta^B + \cdots , \tag{1.4}$$

where the hat indicates the value of a field containing fermionic collective coordinates induced by the instanton background. In this expression the fermionic moduli are packaged into the chiral spinor

$$\zeta^A(x) = \eta^A + (x_\mu - x_{0\,\mu})\sigma^\mu \bar{\xi}^A , \tag{1.5}$$

which, up to rescalings, is also a $(1-\gamma_5)$ projection of a Killing spinor of AdS_5 and indicates the holographic connection between the Yang-Mills instanton and the D-instanton of the IIB string theory in $AdS_5 \times S^5$ [16, 17]. The anti self-dual field strength of the instanton in (1.4) is given by

$$F^{a-}_{\mu\nu} = \frac{4\eta^a_{\mu\nu}\rho_0^2}{((x-x_0)^2+\rho_0^2)^2} , \tag{1.6}$$

where $\eta^a_{\mu\nu}$ is the conventional 't Hooft symbol. The $\cdots$ in (1.4) indicates the presence of terms with six or more powers of fermion parameters that arise from iterating the supersymmetry transformations. The fermionic contribution to the vector potential begins with a term that is quartic in fermions

$$\hat{A}_\mu = \frac{1}{4!} \varepsilon_{ABCD} \zeta^A \sigma_{\mu\nu} \zeta^B D^\nu (F^a_{\lambda\kappa} \zeta^C \sigma^{\lambda\kappa} \zeta^D) + \cdots , \tag{1.7}$$

where $\cdots$ indicates terms with eight or more fermion parameters. The terms of higher order in the fermions, which are not displayed in (1.4) and (1.7), involve not only the combination ζ, but further depend on the broken conformal supersymmetry moduli $\bar{\xi}$.[4] The expression $\hat{A}_\mu$ contributes both to the self-dual field strength and the anti self-dual field strength.[5]

[4]For example, there is an extra term of the form $\varepsilon_{ABCD}\zeta^A\sigma_\mu\bar{\xi}^B(F^a_{\lambda\kappa}\zeta^C\sigma^{\lambda\kappa}\zeta^D)$ in (1.7).

[5]We thank S. Vandoren for pointing out that our statement in [14] was incorrect.

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The calculation of the Wilson loop to leading order in $g_{\rm YM}$ involves substituting expressions (1.4) and (1.7) into (1.1). The sixteen fermionic supermoduli integrals are saturated, in principle, by expanding the exponential to extract the terms with sixteen powers of fermionic coordinates. This involves dealing with complicated combinatorics that arises from various powers of $\hat{\varphi}$'s and $\hat{A}$'s. Significant cancellations between the various terms should arise, just as there are in the calculations of the instanton contribution to composite gauge invariant operators [17, 18]. Nevertheless, explicit evaluation of the integral appears to be prohibitively difficult and we will finesse it by making extensive use of the (super)symmetries in a manner that generalizes the purely bosonic case.

In section 4 we will consider the extension of the six-dimensional representation of the bosonic model to the superconformal setting by introducing four four-component Grassmann spinors, Θ^A, that are chiral spinors of $\mathrm{SO}(4,2)$. These fermions will be associated with the Grassmann coordinates of a supercoset that parameterizes the supersymmetries that are broken by an instanton. By means of a more or less standard construction we will obtain a representation of the $\mathrm{SU}(2,2|4)$ superalgebra and its $\mathrm{OSp}(2,2|4)$ subgroup in terms of bosonic and fermionic coordinates belonging to this supercoset. In this way we will represent the supergroups relevant for the Wilson loop calculation in a chiral fashion suitable for instanton computations. Some of the details of this supercoset construction are contained in appendix B.

The integration over the supermoduli, $(x_0, \rho_0, \eta, \bar{\xi})$ will be considered in section 5. Once again we will move the instanton to the centre of the loop by making use of the residual $\mathrm{OSp}(2,2|4)$. This means moving it to the point $x_0 = 0$ and $\eta = \bar{\xi} = 0$. This again allows the integration over the supermoduli to be carried out as if the theory were abelian. In this manner we end up with an expression for the Wilson loop density on the supermoduli space. The form of this density apparently allows the fermionic variables to be eliminated by a change of bosonic integration variables, so that the integral over the Grassmann variables formally vanishes. However, this neglects the fact that the bosonic integral diverges near the boundary of moduli space (the AdS_5 boundary) and has to be regulated. Any regulator necessarily introduces a dependence on the fermions at the boundary.

An ideal regulator would respect the Poincaré symmetries described in section 1.1. This would require a detailed analysis of the instanton contributions in the theory with $\mathrm{SU}(3) \to \mathrm{SU}(2) \times \mathrm{U}(1)$ in the limit of large symmetry breaking, which we have not carried out. Instead, in section 5 we shall simply cut off the integration over the moduli in a manner that does not respect the Poincaré supersymmetries. We can anticipate that breaking supersymmetry in such a manner will lead to a spurious dependence on the cutoff that will have to be subtracted to restore superconformal invariance. After a certain amount of work the fermionic integrations will be performed explicitly, leading to a density on the bosonic moduli space. This has the form of a complicated tensor that contracts generators in the coset $\mathrm{SU}(2,2)/\,\mathrm{SO}(2,2)$ (or $\mathrm{SO}(5,1)/\,\mathrm{SO}(3) \times \mathrm{SO}(2,1)$ in the euclidean theory) acting on the bosonic loop. The bosonic integration has the form of the integral of a total divergence so, using Gauss' law, the result is given by a boundary contribution, as anticipated. Performing the generalized angular momentum algebra with the aid of the algebraic software package REDUCE enables the explicit calculation of the Wilson loop expectation value.

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As expected, the bulk divergences of the bosonic theory do not arise in the supersymmetric case. However, the result of our calculation does have a linear divergence proportional to the perimeter of the loop.[6]

As argued above, this perimeter divergence is a non conformally-invariant artifact of our cutoff procedure. It can be eliminated by absorbing it into a counterterm for the mass parameter of the test particle that defines the loop. This leaves a finite result for the loop expectation value, which is consistent with conformal symmetry. Furthermore, we will see in section 5.3.2 that the expression for the straight Wilson line has a pure linear divergence of the same value per unit length as for the circular loop, but without the subleading finite part. Therefore, the same mass counterterm eliminates the divergence and leads to the vanishing of the instanton contribution to the straight line expectation value as required by the supersymmetry argument in section 1.1.

In the concluding section we will discuss the generalization of these calculations to the gauge group $\mathrm{SU}(N)$ in the semi-classical limit, which is the limit of weak 't Hooft coupling when $N \to \infty$. It would be good to be able to say something about the limit of strong 't Hooft coupling, which is of relevance for comparison with the supergravity description but this is beyond explicit calculation. We will, however, make some comments on the way in which the instanton contributions to the Wilson loop might be reconciled with the $\mathrm{SL}(2,\mathbb{Z})$ Montonen-Olive duality of the $\mathcal{N}=4$ theory and its image in type-IIB string theory in $AdS_5 \times S^5$. The arguments given will be qualitative. We will also describe the calculation of the instanton contribution to the correlation functions of the Wilson loop with gauge invariant composite operators in. In fact, these calculations are often easier than the pure Wilson loop calculation and result in manifestly finite expressions. In particular, we will give a rather persuasive and simple argument that $\partial\langle W\rangle/\partial\vartheta \neq 0$, as follows if $\langle W\rangle$ has an instanton contribution.

2. Symmetries of the circular loop

The $\mathcal{N}=4$ supersymmetric Yang-Mills theory has a superconformally invariant phase in which the scalar field expectation values are zero. The infinitesimal generators of the four-dimensional conformal group, $(P_\mu, J_{\mu\nu}, D, K_\mu)$, with conjugate parameters $(a^\mu, \omega^{\mu\nu}, \lambda, b^\mu)$, have the following action on the space-time coordinates

$$\delta x^\mu = a^\mu + \omega^{\mu\nu} x_\nu + \lambda x^\mu - x^2 b^\mu + 2b\cdot x x^\mu \,, \tag{2.1}$$

where $x^2 \equiv \eta_{\mu\nu} x^\mu x^\nu$ and $\eta_{\mu\nu} = \mathrm{diag}\,(+---)$. The fifteen transformations of the $\mathrm{SU}(4) \approx \mathrm{SO}(6)$ R-symmetry group (with parameters ω^{ij}) have the form

$$\delta y^i = \omega^{ij} y_j \,. \tag{2.2}$$

In addition, there are four Poincaré supersymmetries with generators Q^α_A and $\bar{Q}^A_{\dot\alpha}$, as well as four superconformal symmetries with generators $S^{A\alpha}$ and $\bar{S}_{A\dot\alpha}$. These generators form

[6]In the preliminary description of our calculation [14] we incorrectly assumed that this divergence would be absent.

the algebra associated with the supergroup SU$(2,2|4)$, in which the fermionic generators satisfy the relations

$$
\begin{aligned}
[P^\mu, S^A_\alpha] &= \sigma^\mu_{\alpha\dot\alpha}\bar{Q}^{\dot\alpha A}\,, \qquad [K^\mu, Q_{\alpha A}] = \sigma^\mu_{\alpha\dot\alpha}\bar{S}^{\dot\alpha}_A\,,\\
[D, Q_{\alpha A}] &= -\frac{1}{2}Q_{\alpha A}\,, \qquad [D, S^A_\alpha] = +\frac{1}{2}S^A_\alpha\,,\\
\{Q_{\alpha A}, \bar{Q}^B_{\dot\alpha}\} &= 2\delta^B_A\sigma^\mu_{\alpha\dot\alpha}P_\mu\,, \qquad \{S^A_\alpha, \bar{S}_{\dot\alpha B}\} = 2\delta^A_B\sigma^\mu_{\alpha\dot\alpha}K_\mu\,,\\
\{\bar{Q}^A_{\dot\alpha}, S^B_\alpha\} &= 0\,, \qquad \{Q_{\alpha\, A}, Q_{\beta\, B}\} = 0\,, \qquad \{S^A_\alpha, S^B_\beta\} = 0\,,\\
\{Q_{\alpha A}, S^{\beta\, B}\} &= \frac{1}{2}\delta_A{}^B(\sigma^{\mu\nu})^\beta_\alpha J_{\mu\nu} + 2\delta^B_A\delta^\beta_\alpha D + 2\delta^\beta_\alpha T^B_A\,.
\end{aligned}
\tag{2.3}
$$

In addition, Q, $\bar{Q}$, S and $\bar{S}$ transform as SO$(3,1)$ spinors of the appropriate chirality and as $\mathbf{4}$'s or $\bar{\mathbf{4}}$'s of SU(4). There is also a central U(1) generator that acts trivially on the elementary fields and local composite operators formed from them.

We are now interested in determining the subgroup of SU$(2,2|4)$ that leaves the circular loop, defined by

$$
(x^1)^2 + (x^2)^2 = R^2\,, \qquad x^3(s) = 0\,, \qquad x^0(s) = 0\,, \qquad \dot{y}^i(s) = |\dot{x}|n^i\,, \tag{2.4}
$$

invariant up to reparametrizations. Since we have fixed a direction (n^i) in the internal space, the loop is only invariant under an SO(5) $\approx$ Sp(4) subgroup of the SU(4) $\approx$ SO(6) R-symmetry group. It is convenient to define the Sp(4) singlet

$$
\Omega_{AB} = n_i\widehat{\Gamma}^i_{AB}\,. \tag{2.5}
$$

The coordinates in the plane of the loop will be denoted by $(x^l) = (x,y) \equiv (x^1, x^2)$ and those transverse to the plane by $(x^t) = (z,t) \equiv (x^3, x^0)$. We will also define the quantities $x_l^2 \equiv -x^l\, x_l$ and $x_t^2 \equiv -x^t\, x_t$ so that $x_l^2 \geq 0$ and (in the Wick-rotated theory) $x_t^2 \geq 0$. The action of infinitesimal SO$(4,2)$ transformations on these coordinates is

$$
\begin{aligned}
\delta x^l &= a^l + \omega^{lm}x_m + \omega^{lt}x_t + \lambda x^l + (x_l^2 + x_t^2)b^l + 2(b^m x_m + b^t x_t)x^l\,,\\
\delta x^t &= a^t + \omega^{tm}x_m + \omega^{ts}x_s + \lambda x^t + (x_l^2 + x_t^2)b^t + 2(b^l x_l + b^s x_s)x^t\,.
\end{aligned}
\tag{2.6}
$$

We are looking for the subset of these transformations that preserves the loop at $x_l^2 = R^2$, $x^t = 0$. Along the loop the transformations in (2.6) become

$$
\begin{aligned}
\delta x^l &= a^l + \omega^{lm}x_m + \lambda x^l + R^2 b^l + 2b^m x_m x^l\,,\\
\delta x^t &= a^t + \omega^{tl}x_l + R^2 b^t\,.
\end{aligned}
\tag{2.7}
$$

The condition $\delta x^t = 0$ implies

$$
a^t = -R^2 b^t\,, \qquad \omega^{tl} = 0\,. \tag{2.8}
$$

The condition $x_l\delta x^l = 0$ is imposed along the loop by contracting the first equation in (2.7) with x_l, giving

$$
0 = a^l x_l - \lambda R^2 - b^l x_l R^2\,, \tag{2.9}
$$

which implies

$$a^l = R^2 b^l\,, \qquad \lambda = 0\,. \tag{2.10}$$

Both ω^{lm} and ω^{ts} remain undetermined. The resulting invariance of the loop is generated by the six generators, J_{xy}, $R^2P_x + K_x$, $R^2P_y + K_y$, J_{zt}, $R^2P_z - K_z$ and $R^2P_t - K_t$. In addition to the obvious rotations in the (x, y) plane and boosts in the (z, t) plane, there are four combinations of translations and conformal boosts. These combinations will be denoted by

$$\Pi_l^+ = RP_l + \frac{1}{R}K_l\,, \qquad \Pi_t^- = RP_t - \frac{1}{R}K_t\,. \tag{2.11}$$

These generators define the algebra $\mathrm{SO}(2,2) = \mathrm{SO}(2,1) \times \mathrm{SO}(2,1)$,

$$\begin{aligned}
[\Pi_x^+, \Pi_y^+] &= J_{xy}\,, \qquad [J_{xy}, \Pi_y^+] = \Pi_x^+\,, \qquad [J_{xy}, \Pi_x^+] = -\Pi_y^+\,,\\
[\Pi_z^-, \Pi_t^-] &= J_{zt}\,, \qquad [J_{zt}, \Pi_t^-] = \Pi_z^-\,, \qquad [J_{zt}, \Pi_t^-] = \Pi_z^-\,,
\end{aligned} \tag{2.12}$$

which is a subalgebra of $\mathrm{SO}(4,2)$. Note that the generators Π_l^- and Π_t^+ are in the coset $\mathrm{SO}(4,2)/\,\mathrm{SO}(2,2)$. The ten $\mathrm{Sp}(4)$ generators that leave the loop invariant are given by the symmetric combinations of the $\mathrm{SU}(4)$ generators

$$T_{AB} = T_A{}^C\Omega_{CB} + T_B{}^C\Omega_{CA}\,. \tag{2.13}$$

The symplectic metric Ω_{AB} and its inverse are used to lower and raise the indices $A, B, \ldots$.

We now want to find the fermionic part of the superconformal group that leaves the loop invariant. Since the bosonic symmetry that preserves the loop, $\mathrm{SO}(2,2) \times \mathrm{Sp}(4)$, is the bosonic part of the supergroup $\mathrm{OSp}(2,2|4)$ (which is a subgroup of $\mathrm{SU}(2,2|4)$) this is a natural candidate for the invariance group of the loop. To verify that this is indeed the case we want to first identify the Killing spinors that satisfy

$$\dot{x}_\mu \sigma^\mu \bar{\kappa}_A(x) = \dot{y}_i \widehat{\Gamma}^i{}_{AB}\kappa^B(x)\,, \tag{2.14}$$

where $\kappa^A = \eta_\oplus^A + x\cdot\sigma\bar{\xi}_\oplus^A$ and $\bar{\kappa}_A = \xi_{\oplus\,A} + x\cdot\bar{\sigma}\bar{\eta}_{\oplus\,A}$ with $\eta^A_{\oplus\,\alpha}$, $\bar{\xi}^A_{\oplus\dot{\alpha}}$, $\bar{\eta}_{\oplus\,A}$ and $\xi_{\oplus\,A}$ constant complex 2-component spinors. Using the parametrization $\dot{x}^l = \omega\varepsilon^{lm}x_m$ and $\dot{y}_i\widehat{\Gamma}^i{}_{AB} = \omega R\Omega_{AB}$ and taking into account that $x^t = 0$ along the loop gives

$$\varepsilon^{lm}x_m\sigma_l(\bar{\eta}_{\oplus\,A} + x_n\bar{\sigma}^n\xi_{\oplus A}) = R\Omega_{AB}(\eta_\oplus^B + x_m\sigma^m\bar{\xi}_\oplus^B)\,. \tag{2.15}$$

More explicitly,

$$(x_1\sigma_2 - x_2\sigma_1)[\bar{\eta}_{\oplus\,A} + (x_1\sigma_1 + x_2\sigma_2)\xi_{\oplus\,A}] = R\Omega_{AB}[\eta_\oplus^B + (x_1\sigma_1 + x_2\sigma_2)\bar{\xi}_\oplus^B]\,. \tag{2.16}$$

Using $x_1^2 + x_2^2 = R^2$, this implies

$$(x_1\sigma_2 - x_2\sigma_1)\bar{\eta}_{\oplus\,A} + R^2\sigma_2\sigma_1\xi_{\oplus\,A} = R\Omega_{AB}[\eta_\oplus^B + (x_1\sigma_1 + x_2\sigma_2)\bar{\xi}_\oplus^B]\,, \tag{2.17}$$

so that

$$\xi_{\oplus\,A} = \frac{1}{R}\Omega_{AB}\sigma^{12}\eta_\oplus^B\,, \qquad \bar{\xi}_\oplus^A = \frac{1}{R}\Omega^{AB}\bar{\sigma}^{12}\bar{\eta}_{\oplus\,B}\,. \tag{2.18}$$

The resulting fermionic symmetries are thus generated by

$$G_A = \sqrt{R}\sigma^{12}Q_A + \frac{1}{\sqrt{R}}\Omega_{AB}S^B\,, \qquad \bar{G}^A = \sqrt{R}\bar{\sigma}^{12}\bar{Q}^A + \frac{1}{\sqrt{R}}\Omega^{AB}\bar{S}_B\,. \tag{2.19}$$

The anti-commutators of the supersymmetry generators are

$$\begin{aligned}\{G_A, \bar{G}_B\} &= R\sigma^\mu P_\mu + \sigma^{12}\sigma^\mu\sigma^{12}\frac{1}{R}K_\mu = \sigma^l\Pi_l^+ + \sigma^t\Pi_t^-\,,\\ \{G_A, G_B\} &= \Omega_{AB}(\sigma^{12}\sigma^{\mu\nu}\sigma^{12} + \sigma^{\mu\nu})J_{\mu\nu} + (T_{AB} + T_{BA})\\ &= 2\Omega_{AB}(\sigma^{xy}J_{xy} + \sigma^{zt}J_{zt}) + (T_{AB} + T_{BA})\,.\end{aligned} \tag{2.20}$$

In a more compact notation the surviving supersymmetry generators G and $\bar{G}$ can be packaged into G^a_A, where a is an index of the $(\mathbf{2},\mathbf{2})$ of $\mathrm{SO}(2,2)$, in which case the supersymmetry algebra reads

$$\{G^a_A, G^b_B\} = \Omega_{AB}J^{ab} + T_{(AB)}\,, \tag{2.21}$$

where the six generators of $\mathrm{SO}(2,2)$, $(\Pi_l^+, J_{xy}, \Pi_t^-, J_{zt})$ have been assembled into J^{ab}. The remaining commutation relations of the $\mathrm{OSp}(2,2|4)$ algebra are

$$\begin{aligned}[J_{ab}, J_{cd}] &= H_{bc}J_{ad} + \text{perms}\,, \qquad [T_{AB}, T_{CD}] = \Omega_{BC}T_{AD} + \text{perms}\,, \qquad [T_{AB}, J_{ab}] = 0\,,\\ [J_{ab}, G_{Ac}] &= H_{bc}G_{Aa} - H_{ac}G_{Ab}\,, \qquad [T_{AB}, G_{Ca}] = \Omega_{BC}G_{Ac} + \Omega_{AC}G_{Bc}\,,\end{aligned} \tag{2.22}$$

where H (to be read as "capital η") denotes the $\mathrm{SO}(2,2)$ invariant metric tensor.

3. One-instanton contribution in the bosonic model

Before tackling the complete integral over the bosonic and fermionic instanton moduli in the $\mathcal{N} = 4$ theory, we will consider some essential features that arise purely from the bosonic integrations. The expression for the Wilson loop that is obtained by simply substituting the BPST instanton solution into (1.1), setting the fermionic variables to zero and ignoring the fermionic integrations will be referred to as the "bosonic model" (it is *not* the Wilson loop of pure Yang-Mills, which has a different and non conformally-invariant measure).

In calculating the Wilson loop we will make use of the fact that the form of the instanton profile is invariant under euclidean conformal transformations with the understanding that the moduli are transformed by compensating conformal transformations. In particular, it is possible to transform a one-instanton configuration into an equivalent one by acting with an element of $\mathrm{SO}(3)\times\mathrm{SO}(2,1)$, which maps the loop onto itself. Since the moduli space is the five-dimensional anti de-Sitter space spanned by (x_0^μ, ρ_0), we will need to represent the action of $\mathrm{SO}(5,1)$ and $\mathrm{SO}(3)\times\mathrm{SO}(2,1)$ on these coordinates. These groups act non linearly on euclidean AdS_5 so it is convenient to represent them in terms of a six-dimensional space with one time-like coordinate which has signature $(5,1)$ on which the groups act linearly. This is the procedure first introduced by Dirac [15] in order to describe the action of $\mathrm{SO}(4,2)$ on four-dimensional Minkowski space.

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3.1 Six-dimensional representation of SO(4,2) and SO(2,2)

The flat six-dimensional coordinates X_M (where X_0 and X_4 are time-like) are taken to satisfy the rotationally invariant constraint

$$X^2 \equiv \eta_{MN} X^M X^N \equiv \eta_{\mu\nu} X^\mu X^\nu + (X_4)^2 - (X_5)^2 = \ell^2 \,, \tag{3.1}$$

where ℓ is a constant scale and for the moment we are using six-dimensional metric $\eta_{MN} = \mathrm{diag}(+ - - - + -)$ appropriate to Minkowski signature in four dimensions $M = 0, 1, 2, 3$. The euclidean case is obtained by the Wick rotation $X_0 \to iX_0$. The constraint is solved in terms of five-dimensional coordinates that parameterize AdS_5 with scale ℓ. A conventional parameterization of AdS_5 in terms of x^μ, ρ, is obtained by the identifications

$$X^\mu = \ell \frac{x^\mu}{\rho} \,, \qquad X^4 = \frac{1}{2}\left(\rho + \frac{\ell^2 - x^2}{\rho}\right) , \qquad X^5 = \frac{1}{2}\left(\rho - \frac{\ell^2 + x^2}{\rho}\right) , \tag{3.2}$$

which represents an AdS_5 hypersurface in $\mathbb{R}^6$. Inverting these conditions gives

$$x^\mu = \ell \frac{X^\mu}{X^4 - X^5} \,, \qquad \rho = \frac{\ell^2}{X^4 - X^5} \,, \qquad \rho^2 - x^2 = \ell^2 \frac{X^4 + X^5}{X^4 - X^5} \,. \tag{3.3}$$

Recall that in the above parametrization the boundary of AdS_5 is at $\rho = 0$. In the following we will also find it useful to define $X^\pm = X^4 \pm X^5$.

It is then easy to check that the Lorentz transformations on the six-dimensional coordinates, generated by

$$L_{MN} = X_M \partial_N - X_N \partial_M \,, \tag{3.4}$$

induce SO(4,2) transformations on AdS_5, with the identifications of the fifteen generators

$$J_{\mu\nu} = L_{\mu\nu} \,, \qquad D = L_{45} \,, \qquad \ell P_\mu = L_{4\mu} + L_{5\mu} \,, \qquad K_\mu = \ell(L_{4\mu} - L_{5\mu}) \,. \tag{3.5}$$

Furthermore the trivial six-dimensional integration measure is equivalent, after the constraint, to the AdS_5 measure

$$\ell^{-4} \int \delta(X^2 - \ell^2)\, d^6 X = \int \frac{d^4 x\, d\rho}{\rho^5} \,. \tag{3.6}$$

The boundary is mapped into itself under the SO(4,2) transformations and the familiar four-dimensional action of SO(4,2) results from the boundary limit in which $\rho \to 0$ [15]. The generators $\Pi^\pm_\mu$ introduced earlier are expressed in this six-dimensional notation as

$$\Pi^\pm_\mu = \left(\frac{\ell}{R} \pm \frac{R}{\ell}\right) L_{4\mu} + \left(\frac{\ell}{R} \mp \frac{R}{\ell}\right) L_{5\mu} \,. \tag{3.7}$$

Our aim is to consider a loop of radius R and to identify AdS_5 with the one-instanton moduli space. For this purpose we will need to consider the euclidean theory obtained by Wick rotation, which involves insertion of judicious factors of i. Since the scale ℓ drops out of all physical quantities it is convenient to choose $\ell = R$ for most of the following (conformal invariance further implies that $\langle W \rangle$ is a constant, independent of R). In this case the expressions (3.7) for $\Pi^{\pm}_{\mu}$ become particularly simple and the stability group generator Π^{+}_{l} acts only on $(X_L) = (X_4, X_l)$ while Π^{-}_{t} acts only on $(X_T) = (X_5, X_t)$. We saw earlier that the loop at $\rho = 0$, $|x_l|^2 = R^2$, $x^t = 0$ is invariant (after a Wick rotation) under $\mathrm{SO}(3)_L \times \mathrm{SO}(2,1)_T$ transformations. These transformations are described in the six-dimensional formalism as those that leave invariant the quadratic form

$$U = X_L^2 \equiv (X_4)^2 - (X_l)^2 = \frac{1}{4}\left(\rho + \frac{R^2 + x_l^2 + x_t^2}{\rho}\right)^2 - R^2\,\frac{x_l^2}{\rho^2}\,, \tag{3.8}$$

where $X_L = (X_1, X_2, X_4)$ are the components of X_M that are transformed by the $\mathrm{SO}(3)_L$ subgroup while $X_T = (X_0, X_3, X_5)$ transform under the $\mathrm{SO}(2,1)_T$ subgroup.[7]

For fixed U equation (3.8) defines a four-dimensional hyperbolic surface in AdS_5, once the constraint (3.1) is imposed. Such four-surfaces of constant U foliate the interior of AdS_5 in such a manner that they all meet on a circle of radius R centered on the point $x^\mu = 0$ on the boundary at $\rho = 0$. In other words, all the four-surfaces are bounded by the Wilson loop if the boundary of AdS_5 is identified with four-dimensional space-time (see the figure). The value of U is in the range $R^2 \leq U \leq \infty$. There are two surfaces for every value of $U > R^2$, while the surface with the minimal value, $U = R^2$, is degenerate since it is two dimensional. It is defined by $x_t = 0$, $x_l^2 + \rho^2 = R^2$, which is just the surface of minimal area embedded in AdS_5 which bounds the loop of radius R on the boundary and was considered in [6].

3.2 The Wilson loop in the bosonic model

The Wilson loop expectation value in the toy bosonic model is given by

$$\langle W_B \rangle = \int \frac{d^4x_0 d\rho_0}{\rho_0^5}\, W_B[x(\cdot); x_0, \rho_0]\,, \tag{3.10}$$

where

$$W_B[x(\cdot); x_0, \rho_0] = \frac{1}{2}\,\mathrm{Tr}\,\mathcal{P}\, e^{i\int_C A\cdot\dot{x}}\,, \tag{3.11}$$

with A denoting the standard BPST instanton solution, and where we are temporarily adopting a notation that makes explicit that W_B is a functional of the points on a circle on the boundary of AdS_5.

It will be important that W_B is invariant under arbitrary $\mathrm{SO}(3) \times \mathrm{SO}(2,1)$ transformations that map the circle into itself (in euclidean signature). To show this, consider the action of a general $\mathrm{SO}(3) \times \mathrm{SO}(2,1)$ transformation, $x \to \gamma(x) \equiv \tilde{x}$ on $W_B[x(\cdot); x_0, \rho_0]$. On the one hand, this transformation maps the circle into itself so that, after a reparameterization of the circle (under which W_B is invariant), the points on the circle are not transformed

[7] For convenience we are using the conventions

$$\begin{aligned} X_L^2 &= (X_4)^2 - (X_1)^2 - (X_2)^2 \equiv (X_4)^2 - X_l^2\,,\\ X_T^2 &\equiv \mp(X_0)^2 + (X_3)^2 + (X_5)^2 \equiv (X_5)^2 + X_t^2\,, \end{aligned} \tag{3.9}$$

with the $-$ sign for Minkowski signature and $+$ sign after the Wick rotation of the time coordinate. This means that with euclidean signature $X_t^2, X_T^2 \geq 0$ and therefore $X_L^2 = X_T^2 + R^2 \geq R^2$ after using the constraint.

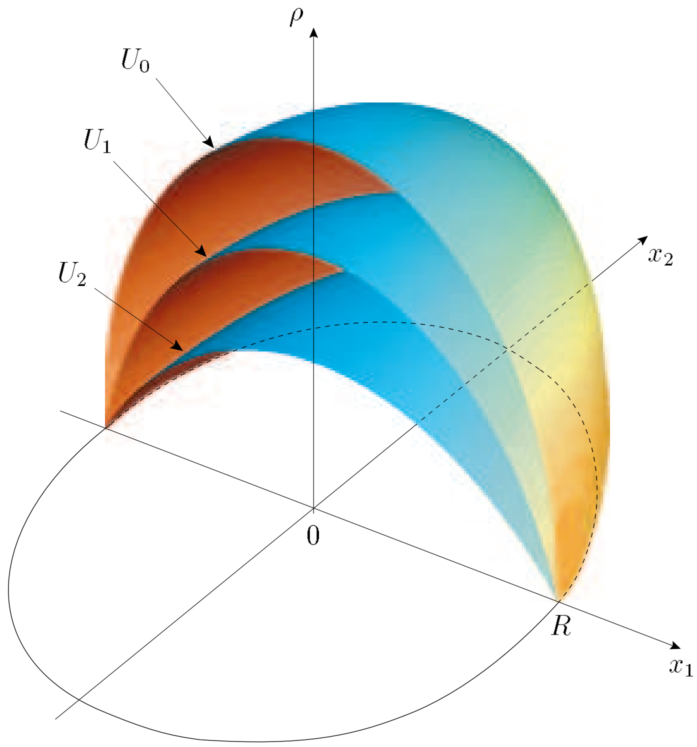

Figure 1: Surfaces of constant U are SO(2, 2) orbits of codimension 1 in AdS_5. The plot shows a section with $x_t = 0$. All surfaces with constant U end on the loop $x_l^2 = R^2$, $x_t = 0$ on the boundary, $\rho = 0$. The surface with the minimal value, $U = U_0 = R^2$, coincides with the minimal two-dimensional surface bounded by the loop. The surfaces shown have $U \geq R^2$ and intersect the ρ axis with $\rho \leq R$. The surfaces that intersect the axis with $\rho > R$ have been omitted from the figure.

and therefore $W_B[x(\cdot); x_0, \rho_0] = W_B[\tilde{x}(\cdot); x_0, \rho_0]$. On the other hand the transformation of x in the instanton solution is equivalent (up to an irrelevant gauge transformation) to a transformation on the instanton moduli $(x_0, \rho_0) \to (\tilde{x}_0, \tilde{\rho}_0) = \gamma^{-1}(x_0, \rho_0)$. Therefore

$$W_B[x(\cdot); x_0, \rho_0] = W_B[x(\cdot); \tilde{x}_0, \tilde{\rho}_0]\,, \tag{3.12}$$

and so the density W_B depends only on the choice of SO(3) × SO(2, 1) orbit, which can be labeled by the value of the invariant, $U = X_L^2$, defined in the previous subsection.

We will soon find it useful to choose an appropriate SO(2, 2) transformation, γ, that moves the instanton to a point $(\tilde{x}_0^\mu = 0, \tilde{\rho}_0)$, which is at the centre of the loop. The scale of the transformed instanton is fixed by the invariance of $U = X_L^2 = X_L^2(\tilde{x}_0 = 0; \tilde{\rho}_0)$, which implies

$$\frac{1}{4}\left(\rho_0 + \frac{R^2 + x_l^2 + x_t^2}{\rho_0}\right)^2 - R^2\frac{x_l^2}{\rho_0^2} = \frac{1}{4}\left(\frac{R^2}{\tilde{\rho}_0} + \tilde{\rho}_0\right)^2, \tag{3.13}$$

or

$$\tilde{\rho}_0 = |X_L| - |X_T|\,. \tag{3.14}$$

This expression relates the parameters of an instanton at a generic position in moduli space to one at $\tilde{x}_0 = 0$ with a scale $\tilde{\rho}_0(x^\mu, \rho)$. The explicit transformation that moves the

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instanton from one point to another along an $\mathrm{SO}(3)\times\mathrm{SO}(2,1)$ orbit is a group element of the form $\exp(a^l\Pi_l^+ + a^t\Pi_t^-)$, where the parameters $a^l(x_0^\mu,\rho_0)$ and $a^t(x_0^\mu,\rho_0)$ are specific functions of the collective coordinates (which we do not need in the following).

It will prove efficient to express the integral over the instanton moduli space, (3.10), as a six-dimensional integral with flat measure together with a δ-function constraint

$$\begin{aligned}\langle W_B\rangle &= \frac{1}{R^4}\int d^6X_0\,\delta(X_0^2-R^2)\,\frac{1}{2}\,\mathrm{Tr}\,\mathcal{P}\,e^{i\int_C A\cdot\dot{x}ds}\\ &= \frac{1}{R^4}\int d^6X_0\,\delta(X_0^2-R^2)\,W_B(X_0)\,,\end{aligned} \tag{3.15}$$

where

$$W_B(X_0)=\frac{1}{2}\,\mathrm{Tr}\,P\,\exp\left(i\int\frac{\eta^a_{\mu\nu}(x^\mu-x_0^\mu)\sigma_a}{\rho_0^2+(x-x_0)^2}\,\dot{x}^\nu\,ds\right). \tag{3.16}$$

The standard one-instanton solution in a "non-singular gauge" (a gauge in which the singularity arises at $|x|=\infty$) has been substituted for A_ν in (3.16). The Pauli matrices σ^a describe the SU(2) colour symmetry. Expression (3.15) does not include the correct prefactor that arises from gaussian fluctuations, which we are ignoring in the bosonic model.[8]

In the special case in which the instanton is at the centre of the loop the path ordered exponential simplifies since $\sigma_a\eta^a_{\mu\nu}\,x^\mu\,\dot{x}^\nu ds = R^2\sigma^3 d\phi$, where $0\le\phi\le 2\pi$ is the angle around the loop, which has been taken to lie in the (x^1,x^2) plane. So in this special configuration the exponent is proportional to σ^3 and the path ordering becomes trivial, as in the abelian theory. More generally, it should always be possible to choose a gauge in which the connection along a given curve is a non-vanishing constant (analogous comments concerning the maximally abelian gauge appear in [19, 20, 21]). The integration over the angle ϕ followed by evaluation of the trace leads to (dropping the subscript 0)

$$W_B(X)=\cos\left(\frac{2\pi R^2}{R^2+\tilde{\rho}^2}\right). \tag{3.17}$$

The value of $\tilde{\rho}$ may be expressed in terms of the invariant $X_T^2=U-R^2$ by using (3.14), giving

$$\frac{2\pi R^2}{R^2+\tilde{\rho}^2}=\pi+\frac{\pi|X_T|}{\sqrt{X_T^2+R^2}}\,, \tag{3.18}$$

so the Wilson loop density can be expressed as

$$W_B(X)=-\cos\left(\frac{\pi|X_T|}{\sqrt{X_T^2+R^2}}\right). \tag{3.19}$$

Since W_B depends only on one parameter $|X_T|=\sqrt{U-R^2}$ that labels the $\mathrm{SO}(3)\times\mathrm{SO}(2,1)$ orbits, the five-dimensional integration over the bosonic moduli in (3.15) reduces to a one-dimensional integral over $|X_T|$ with a measure that is proportional to the volume

[8] Various overall numerical constants will be dropped from the expressions for the Wilson loop but they will be reinstated in the final result.

of the orbit labeled by $|X_T|$. This leads to an infinite value due to the divergence of the SO$(2,1)$ volume near the boundary of moduli space ($\rho = 0$ in our five-dimensional coordinates). Similar considerations will also apply to the $\mathcal{N} = 4$ theory and we therefore anticipate the need to introduce a regulator that suppresses point-like instantons. Other regions of moduli space also lead to divergences in the bosonic model.

We will proceed by explicitly performing the two-dimensional X_l integration in (3.15) to eliminate the δ function and regulating the remaining integrations by introducing a large X_4 cutoff,

$$X_4 \leq \Lambda \equiv \frac{R^2}{\epsilon}, \tag{3.20}$$

where ϵ is a small scale with dimensions of length. This cutoff manifestly preserves a SO(5) subgroup of SO$(5,1)$. Performing the X_4 integral gives

$$\begin{aligned} \langle W_B \rangle_\epsilon &= -\frac{2\pi}{R^4} \int_{|X_T| \leq \sqrt{\Lambda^2 - R^2}} d^3 X_T \left(\Lambda - \sqrt{X_T^2 + R^2} \right) \cos\left(\frac{\pi |X_T|}{\sqrt{X_T^2 + R^2}} \right) \\ &= -2\pi \int_{|\mathbf{X}| \leq \sqrt{R^2/\epsilon^2 - 1}} d^3 \mathbf{X} \left(\frac{R}{\epsilon} - \sqrt{|\mathbf{X}|^2 + 1} \right) \cos\left(\frac{\pi |\mathbf{X}|}{\sqrt{|\mathbf{X}|^2 + 1}} \right), \end{aligned} \tag{3.21}$$

where $\mathbf{X} \equiv X_T/R$. This integral diverges when $\epsilon \to 0$. The leading divergence is of order ϵ^{-4} and arises from the regions in which the instanton has a scale that is very much greater than R and those in which it has a fixed scale but is very far from the loop. This bulk divergence has no analogue in the $\mathcal{N} = 4$ case to be considered later. The region where the instanton scale is very much smaller than R leads to the divergence of order ϵ^{-1} associated with the volume of the SO$(3) \times$SO$(2,1)$ orbits and which will remain as an important issue in the analysis of the supersymmetric theory.

3.3 Comments on the straight Wilson line

If the curve $\mathcal{C}$ is taken to be a straight line there is room for potential confusion. One well-defined way to obtain a straight Wilson line is to consider it to be a stereographic projection of a circular loop passing through the north pole of S^4 that is the boundary of euclidean AdS_5. Since the north pole is not a special point on the sphere, in a conformally invariant theory this gives a result that is the same as that for a generic circular loop. However, this differs from the natural definition of a straight Wilson line defined directly in $\mathbb{R}^4$. The latter corresponds to the starting point of [11] where it was emphasized that the presence of a conformal anomaly leads to a different expression from the circular loop.[9] This will also be true in the presence of an instanton.

One subtlety in the analysis of the instanton contribution to the straight line concerns the gauge independence of the calculation. The BPST instanton solution for the gauge potential in the so-called "non-singular" gauge is, in fact, singular at the point at infinity. Since this point coincides with a point on the straight line it is best to avoid this gauge

[9]One way to think of this straight Wilson line is to consider it to be the limit of a "thermal Wilson loop", or a Polyakov loop, defined in $\mathbb{R}^3 \times S^1$, where the circular dimension has infinite radius.

and use a "singular" gauge, in which the gauge potential is singular at the point $\mathbf{x} = \mathbf{x_0}$, $t = t_0$. This gives the expression

$$\langle W_B(\text{line})\rangle = \int \frac{d^3\mathbf{x}_0 dt_0 d\rho_0}{\rho_0^5}\, W_B^s\,, \tag{3.22}$$

for the expectation value of the Wilson line at $x^i = 0$, where

$$W_B^s = \frac{1}{2}\,\mathrm{Tr}\,\mathcal{P}\left(\exp i\int_{-\infty}^{+\infty} \frac{\rho_0^2\,\mathbf{x}_0\cdot\sigma dt}{[\rho_0^2 + |\mathbf{x}_0|^2 + (t-t_0)^2][|\mathbf{x}_0|^2 + (t-t_0)^2]}\right). \tag{3.23}$$

We have used the fact that the vector potential is proportional to $\eta^a_{0i}(x^i - x^i_0)\sigma_a$ where $\eta^a_{01} = \delta^a_i$. The SU(2) connection in this expression is abelian since it always points along $\mathbf{x}_0\cdot\sigma$ so the path ordering is immaterial and the integral is then easily evaluated giving

$$\int_{-\infty}^{+\infty} \frac{\rho_0^2 dt}{[\rho_0^2 + |\mathbf{x}_0|^2 + (t-t_0)^2][|\mathbf{x}_0|^2 + (t-t_0)^2]} = \frac{\pi}{|\mathbf{x}_0|} - \frac{\pi}{\sqrt{|\mathbf{x}_0|^2 + \rho_0^2}}\,, \tag{3.24}$$

so that

$$W_B^s = \frac{1}{2}\,\mathrm{Tr}\exp\left(\frac{i\pi\mathbf{x}_0\cdot\sigma}{|\mathbf{x}_0|}\left[1 - \frac{|\mathbf{x}_0|}{\sqrt{|\mathbf{x}_0|^2+\rho_0^2}}\right]\right) = -\cos\left(\frac{\pi|\mathbf{x}_0|}{\sqrt{\rho_0^2 + |\mathbf{x}_0|^2}}\right). \tag{3.25}$$

Starting, instead, with the instanton in the "non-singular" gauge gives a result for the Wilson loop expectation value with the opposite sign. However in that case there is a subtlety because the gauge potential is actually singular at infinity, which is a point on the straight line. To avoid this problem, in the following we shall choose the connection in the "singular" gauge although the final results should not depend on this choice. Notice that in the case of the straight line the calculation could also be simplified working in the Weyl gauge [22].

The expression (3.25) has the same structure as (3.19), which refers to circular loops, but with $|X_T|/R$ replaced by $|\mathbf{x}_0|/\rho_0$. However, this is a rather formal correspondence since in both cases integration over the moduli gives a divergent result. In order to understand the connection between the straight line and the circular Wilson loops more quantitatively we will discuss the group theoretical relation between them. In euclidean signature the stability group of the straight line and the circle are isomorphic subgroups of SO(5, 1) — they are both SO(3) × SO(2, 1). One of these subgroups is obtained from the other by conjugation with an infinite boost generated by Π_l^-. Alternatively, the straight line may be obtained from the circle by inversion with respect to a point on the circle. In the case of the straight line the SO(3) factor refers to rotations in the three dimensions orthogonal to the line and the SO(2, 1) subgroup corresponds to a dilation combined with a translation and a conformal boost along the direction of the line. In the supersymmetric case the surviving R-symmetry is Sp(4) ≈ SO(5), as for the circle. There is, however, a crucial difference between the straight line and the circle when a regulator is introduced. In terms of the AdS_5 description of the regulated field theory the natural regulator for the straight line defined in $\mathbb{R}^4$ is the Poincaré invariant condition $\rho_0 \geq \epsilon$ which preserves the isometries of the loop. In this case the expectation value of the Wilson line has a finite contribution per unit length, but diverges due to the integration over the length. Upon mapping the infinite line to a circle through the north pole of the S^4 boundary of B^5 this cutoff prescription is pathological since the $\rho = \epsilon$ surface touches the boundary at the north pole. It is therefore not a good regulator for loops on S^4. An appropriate regulator in this case is obtained by cutting off the moduli integrations on a spherical shell inside the B^5 boundary in a manner that preserves SO(5). As we will see, in the superconformal case this distinction becomes of paramount importance since supersymmetry implies that the expectation value is unity in the case of the straight Wilson line while it has non-trivial dependence on the coupling for a circular loop.

The expression for the straight Wilson line in the presence of a $\rho \geq \epsilon$ cutoff has the form

$$\begin{aligned}\langle W_B(\text{line})\rangle_\epsilon &= \int dt_0 \int d^3\mathbf{x}_0 \int_\epsilon^\infty \frac{d\rho_0}{\rho_0^5}\, W_B^s = -\int dt_0 \int_\epsilon^\infty \frac{d\rho_0}{\rho_0^2} \int d^3\mathbf{X}\,\cos\left(\frac{\pi|\mathbf{X}|}{\sqrt{|\mathbf{X}|^2+1}}\right)\\ &= -\int dt_0\, \frac{1}{\epsilon}\int d^3\mathbf{X}\,\cos\left(\frac{\pi|\mathbf{X}|}{\sqrt{|\mathbf{X}|^2+1}}\right),\end{aligned} \tag{3.26}$$

where $\mathbf{X} = \mathbf{x}_0/\rho_0$. This is the integral of a constant density over the infinite length of the line. In the limit $\epsilon \to 0$ the divergence in the density is *identical* to that arising from the $1/\epsilon$ term in the last equation in (3.21) that describes a circular loop of length $2\pi R$. As remarked earlier, the three-dimensional $\mathbf{X}$ integration gives rise to the bulk divergences of the bosonic model which are not present in the supersymmetric theory.

4. Instanton superspace and the $\mathcal{N} = 4$ Wilson loop

The bosonic toy model is not a realistic description of any bosonic quantum field theory. In the case of pure Yang-Mills the quantum measure of integration involves a ratio of determinants which is not scale invariant. A detailed computation of these determinants was performed in 't Hooft's seminal paper on instanton calculus [23]. In supersymmetric theories the fluctuation determinant leads to a dependence on the scale parameter μ of the form

$$(\mu\rho)^{n_B - \frac{1}{2}n_F}\,, \tag{4.1}$$

where n_B and n_F are the numbers of bosonic and fermionic zero modes, respectively. The exponent $n_B - \frac{1}{2}n_F = k\beta_1$ in (4.1) is the coefficient of the β-function of $g_{\rm YM}$ at one-loop times the instanton number. Indeed for a generic $\mathcal{N} = 1$ supersymmetric theory

$$\beta_1 = \left(\frac{11}{3} - \frac{1}{2}\times\frac{4}{3}\right) C_A - \sum_R \left(\frac{1}{2}\times\frac{4}{3} + \frac{2}{6}\right) C_R\,, \tag{4.2}$$

where the C_A term comes from the vector supermultiplet and the C_R terms comes from a sum of the chiral supermultiplets in representations labelled R (C_A and C_R being the appropriately normalized Dynkin indices). Index theorems imply that

$$n_B = 4kC_A\,, \qquad n_F = 2kC_A + 2k\sum_R C_R\,. \tag{4.3}$$

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For (super-)conformal theories, such as $\mathcal{N}=4$ Yang-Mills, $\beta=0$ and the quantum measure for the collective (super-)coordinates exactly coincides with the classical one. For supersymmetric non-superconformal theories the quantum measure is deformed so it is "squashed". But, in principle, one can still keep track of the transformations under the sequence of steps that bring the instanton to the centre of the circular loop. Although this is not the subject of the current paper it should be possible to generalize the manipulations of the superconformal $\mathcal{N}=4$ theory to these cases.

In order to extend the analysis of section 3 to $\mathcal{N}=4$ supersymmetric Yang-Mills theory we need to include the sixteen fermionic collective coordinates, η^A_α and $\bar\xi^A_{\dot\alpha}$. This means that we need to consider the extension of the six-dimensional representations of SO(4,2) and SO(2,2) (and their euclidean continuations) to the supersymmetric case.

4.1 Six-dimensional chiral representation of SU(2,2|4) and OSp(2,2|4)

Using the methods of [24] we will present a supercoset construction to represent the action of the superconformal group SU(2,2|4) in terms of the six bosonic coordinates X^M and of four Grassmann spinors, Θ^A_a, where a is a four-component spinor index appropriate to a Weyl spinor in $D=6$ with signature $(4,2)$. This provides a chiral representation of the one-instanton superspace.

The instanton solution breaks a subset of the bosonic and fermionic symmetries in SU(2,2|4). It is invariant under rotations modulo gauge transformations and under the linear combination $P_\mu + K_\mu/\rho^2$ of translational and conformal boost symmetries. Altogether this means that an SO(4,1) subgroup of SO(4,2) is left unbroken. In addition, the instanton preserves the $\mathrm{SU}(4)_R$ symmetry. The supersymmetries that remain unbroken by the instanton are $\bar Q^A_{\dot\alpha}$ and S^A_α. Putting these together means that the $\mathcal{N}=4$ instanton superspace may be described by the coset

$$\frac{G}{H} \equiv \frac{\mathrm{SU}(2,2|4)}{\mathrm{Span}\{\mathrm{SO}(4,1)\times\mathrm{SU}(4);\bar Q^A_{\dot\alpha}, S^B_\alpha\}}\,, \tag{4.4}$$

where the elements of H form the stability group of unbroken generators. Although there is a great deal of ambiguity in the choice of coordinates, it is convenient to choose the coset representative

$$V(x^\mu,\eta^A_\alpha,\bar\xi^A_{\dot\alpha},\lambda) = e^{xP}e^{\eta Q}e^{\bar\xi\bar S}e^{\lambda D}\,, \tag{4.5}$$

with inverse

$$V^{-1}(x^\mu,\eta^A_\alpha,\bar\xi^A_{\dot\alpha},\lambda) = e^{-\lambda D}e^{-\bar\xi\bar S}e^{-\eta Q}e^{-xP}\,. \tag{4.6}$$

For simplicity the subscript 0 has been dropped from the collective coordinates. The left invariant 1-form

$$L = V^{-1}dV = e^{-\lambda}dxP + e^{-\lambda/2}(\bar\xi dx\cdot\bar\sigma + d\eta)Q + e^{\lambda/2}d\bar\xi\bar S + d\lambda D\,, \tag{4.7}$$

satisfies the Maurer-Cartan equation

$$dL - L\wedge L \equiv \left(dL^\Lambda + \frac{1}{2}L^\Delta\wedge L^\Sigma f_{\Sigma\Delta}{}^\Lambda\right)T_\Lambda = 0\,, \tag{4.8}$$

where $f_{\Sigma\Delta}{}^{\Lambda}$ are the structure constants and T_Λ denotes the generators of SU$(2,2|4)$ which divide into those that are in the coset and those that are in the stability group,

$$T_\Lambda = (C_A,\ H_i)\,, \tag{4.9}$$

where A labels the elements of the coset and i the elements of the stability group. The one-form may then be decomposed into the super-vielbein (E_M^A) and H-connection (ω_M^i)

$$L = dZ^M(E_M^A C_A + \omega_M^i H_i) = dZ^M\, L_M^\Lambda T_\Lambda\,, \tag{4.10}$$

where M is the "coordinate index". The components of the super-vielbein follow from (4.7)

$$\begin{aligned} E_\mu^{\hat\mu} &= e^{-\lambda}\delta_\mu^{\hat\mu}\,, \qquad E_\mu^{\hat\alpha\hat A} = e^{-\lambda/2}(\bar\xi\bar\sigma_\mu)^{\alpha\hat A}\,, \qquad E_{\alpha A}^{\hat\alpha\hat A} = e^{-\lambda/2}\delta_\alpha^{\hat\alpha}\delta_A^{\hat A}\,,\\ E_{\dot\alpha A}^{\hat{\dot\alpha}\hat A} &= e^{+\lambda/2}\delta_{\dot\alpha}^{\hat{\dot\alpha}}\delta_A^{\hat A}\,, \qquad E_d^{\hat d} = 1\,, \end{aligned} \tag{4.11}$$

with inverse

$$\begin{aligned} E_{\hat\mu}^{\mu} &= e^{+\lambda}\delta_{\hat\mu}^{\mu}\,, \qquad E_{\hat\mu}^{\alpha A} = -e^{+\lambda/2}(\bar\xi\bar\sigma_\mu)^{\alpha A}\,, \qquad E_{\hat\alpha\hat A}^{\alpha A} = e^{+\lambda/2}\delta_{\hat\alpha}^{\alpha}\delta_{\hat A}^{A}\,,\\ E_{\hat{\dot\alpha}\hat A}^{\dot\alpha A} &= e^{-\lambda/2}\delta_{\hat{\dot\alpha}}^{\dot\alpha}\delta_{\hat A}^{A}\,, \qquad E_{\hat d}^{d} = 1\,. \end{aligned} \tag{4.12}$$

The superisometries of the supercoset,

$$\delta Z^M = -\Xi^M\,, \tag{4.13}$$

are defined to be those transformations of the super-coordinates that satisfy

$$\mathcal{L}_\Xi L + d\Lambda_\Xi^{(H)} + [L, \Lambda_\Xi^{(H)}] = \left(\mathcal{L}_\Xi L^\Lambda + d\Lambda^i\,\delta_i^\Lambda + \Lambda^i\,L^\Sigma\,f_{\Sigma i}{}^{\Lambda}\right)\,T_\Lambda = 0\,, \tag{4.14}$$

where $\mathcal{L}_\Xi$ is the Lie super-derivative. This means that a coordinate transformation along Ξ can be compensated by a local H-transformation, $\Lambda_\Xi^{(H)}$. This equation is not G-covariant and it is convenient to rewrite it in terms of a covariantly constant Killing supervector,

$$\Sigma \equiv \Sigma^\Lambda G_\Lambda \equiv \Sigma^A C_A + \Sigma^i H_i\,, \tag{4.15}$$

which is defined by

$$\Sigma^{\hat A} = \Xi^A\, E_A{}^{\hat A}\,, \qquad \Sigma^i = \Lambda^i + \Xi^M\,\omega_M{}^i\,. \tag{4.16}$$

By virtue of (4.8) and (4.14) this satisfies

$$D\Sigma \equiv d\Sigma + [L,\Sigma] = 0\,. \tag{4.17}$$

This equation has the G-invariant solution

$$\Sigma = V^{-1}\Sigma_\oplus V\,, \tag{4.18}$$

where $\Sigma_\oplus$ is any constant element of the Lie algebra of G. In our case

$$\Sigma_\oplus^A C_A = x_\oplus^\mu P_\mu + \eta_\oplus^{\alpha A} Q_{\alpha A} + \bar\xi_\oplus^{\dot\alpha A}\bar S_{\dot\alpha A} + \lambda_\oplus D\,, \tag{4.19}$$

and

$$\Sigma_\oplus^i H_i = b_\oplus^\mu K_\mu + \frac{1}{2}\omega_\oplus^{\mu\nu}J_{\mu\nu} + \bar\eta_{\oplus\,\dot\alpha A}\bar Q^{\dot\alpha A} + \xi_{\oplus\,\alpha A}S^{\alpha A}\,. \tag{4.20}$$

The explicit expressions for Σ^A are calculated in detail in appendix B. The isometries follow by inverting (4.16),

$$\Xi^M = \Sigma^A E_A^M = (V^{-1}\Sigma_\oplus V)^A E_A^M\,, \tag{4.21}$$

and these are also given explicitly in appendix B.

4.2 Grassmann variables for instanton superspace

The quantities x^μ, $\rho \equiv e^\lambda$, η and $\bar\xi$ are to be identified with the collective super-coordinates of the instanton. The transformations given in appendix B suggest that the fermionic variables should be packaged together into a sixteen component chiral spinor, Θ_a^A, where $a = (\alpha, \dot\alpha)$ is a spinor index of SO(4,2) (or SO(5,1) in euclidean signature). This is achieved by defining

$$\Theta_a^A = (\eta_\alpha^A + x \cdot \sigma_{\alpha\dot\alpha}\bar\xi^{\dot\alpha A}, \bar\xi_{\dot\alpha}^A)\,. \tag{4.22}$$

The chirality of this spinor is defined with the chirality of its four-dimensional spinor components which, in turn, are correlated with the chirality of the BPST instanton solution. The 32 supersymmetry parameters are contained in a spinor ε_a^A and its conjugate $\bar\varepsilon_A^a$, defined by

$$\varepsilon_a^A = (\eta_\oplus^A + \sigma \cdot x\,\bar\xi_\oplus^A\,,\ \ \bar\xi_\oplus^A)\,, \qquad \bar\varepsilon_A^a = (\bar\eta_{\oplus\,A}\,,\ \ \xi_{\oplus\,A} + \sigma\cdot x\,\bar\eta_{\oplus\,A})\,. \tag{4.23}$$

The superconformal transformations on the coordinates x^μ, ρ, η and $\bar\xi$ can be compactly rewritten as

$$\begin{aligned} \delta\Theta_a^A &= \varepsilon_a^A\,, \qquad & \bar\delta\Theta_a^A &= -\Theta_b^A\Theta_a^B\bar\varepsilon_B^b\,,\\ \delta X^M &= 0\,, \qquad & \bar\delta X^M &= \frac{1}{2}\bar\varepsilon_A\Gamma^{MN}\Theta^A X_N\,. \end{aligned} \tag{4.24}$$

The transformations in (4.24) are generated by supercharges $\varepsilon_a^A\mathcal{Q}_A^a$ and $\bar\varepsilon_A^a\bar{\mathcal{Q}}_a^A$, where

$$\mathcal{Q}_A^a = \frac{\partial}{\partial\Theta_a^A}, \qquad \bar{\mathcal{Q}}_a^A = \Theta_b^A\Theta_a^B\frac{\partial}{\partial\Theta_b^B} + \frac{1}{4}\Gamma_a^{MNb}\Theta_b^A L_{MN}\,, \tag{4.25}$$

and satisfy the SU(2,2|4) superalgebra

$$\{\mathcal{Q}_A^a, \bar{\mathcal{Q}}_b^B\} = \frac{1}{4}\delta_A^B\Gamma_b^{MN\,a}J_{MN} + \frac{1}{4}\delta_b^a\widehat\Gamma^{ij}{}_A{}^B T_{ij}\,, \tag{4.26}$$

with $\{\mathcal{Q},\mathcal{Q}\} = \{\bar{\mathcal{Q}},\bar{\mathcal{Q}}\} = 0$, where $J_{MN} = L_{MN} + S_{MN}$ are the standard generators of SO(4,2) and T_{ij} are the generators of SO(6). More explicitly,

$$\begin{aligned} L_{MN} &= X_M\partial_N - X_N\partial_M\,, \qquad S_{MN} = \frac{1}{2}\,\Theta_a^A\Gamma^a_{MNb}\frac{\partial}{\partial\Theta_b^A}\,,\\ T_{ij} &= \frac{1}{2}\,\Theta_a^A\widehat\Gamma_{ijA}{}^B\frac{\partial}{\partial\Theta_a^B}\,. \end{aligned} \tag{4.27}$$

In principle, this algebra can be extended by adding a term $\alpha\Theta$ to $\bar{\mathcal{Q}}$. This generates an additional term $\alpha\,\delta_b^a\,\delta_A^B$ on the right-hand side of (4.26), which is a U(1) central extension. The other anticommutators are unaffected. In what follows we have only been able to make sense of $\langle W\rangle$ by choosing $\alpha = 0$ as in (4.27).

4.3 Superinvariants

The bosonic invariant of SO(4, 2), $R^2 = \eta_{MN}X^M X^N$, is also invariant under SU(2, 2|4), as is easily seen from the representation defined by (4.25) and (4.27). Given an invariant of the bosonic subgroup of OSp(2, 2|4), such as the quadratic invariant $U \equiv X_L^2 = (X_4)^2 - (X_1)^2 - (X_2)^2 = R^2 - (X_3)^2 - (X_0)^2 - (X_5)^2 \equiv R^2 - V$, it is natural to ask whether it is possible to find its supersymmetric extension.

The condition that a quantity Ψ should be invariant under a linear combination of the supersymmetries requires

$$(\varepsilon_a^A \mathcal{Q}_A^a + \bar{\varepsilon}_A^a \bar{\mathcal{Q}}_a^A)\Psi(X,\Theta) = 0\,, \tag{4.28}$$

with the boundary condition $\Psi(X, 0) = \Psi_B(X)$ for some bosonic invariant $\Psi_B(X)$. The restriction to OSp(2, 2|4) is implemented by choosing

$$\bar{\varepsilon}_A^a = \Omega_{AB} H^{ab} \varepsilon_b^B\,, \tag{4.29}$$

where $\Omega_{AB} = n_i \widehat{\Gamma}^i{}_{AB}$ is antisymmetric in A and B and $H^{ab} = \Gamma^{ab}_{412}$ is symmetric in a and b. The former is a symplectic metric of Sp(4) $\approx$ SO(5) while the latter is a symmetric metric of SO(2, 2). Each of these can be used to raise and lower indices of the relevant bosonic subgroup. The condition (4.29) eliminates half of the supersymmetry parameters, so (4.28) ensures invariance under a total of sixteen residual supersymmetries that may be parameterized by ε_a^A. In our case the superinvariant, $\Psi(X, \Theta)$, will be the Wilson loop density in supermoduli space, $W(X, \Theta)$, while $\Psi_B(X)$ will be the bosonic density $W_B(X)$ (defined in (3.19)). Equation (4.28) can be written as

$$\frac{\partial W(X,\Theta)}{\partial \Theta_a^A} + \Omega_{AB} H^{ab}\left[\Theta_c^B \Theta_b^C \frac{\partial W(X,\Theta)}{\partial \Theta_c^C} + \frac{1}{4}\Gamma_b^{MNc}\Theta_c^B L_{MN}\, W(X,\Theta)\right] = 0 \tag{4.30}$$

with $W(X, 0) = W_B(X)$. It is convenient to rewrite this in the form

$$D_{Ab}^{aB}(\Theta)\frac{\partial W(X,\Theta)}{\partial \Theta_b^B} = -\frac{1}{4}\Omega_{AB} H^{ab}\Gamma_b^{MNc}\Theta_c^B L_{MN}\, W(X,\Theta)\,, \tag{4.31}$$

where

$$D_{Ab}^{aB}(\Theta) = \delta_b^a \delta_A^B + \Omega_{AC} H^{ac}\Theta_b^C \Theta_c^B\,. \tag{4.32}$$

Inverting $D_{Ab}^{aB}(\Theta)$ and setting $B(t) = W(X, t\Theta)$ gives

$$\begin{aligned}\frac{dB(t)}{dt} = \frac{\Theta}{t}\frac{\partial W(X,t\Theta)}{\partial \Theta} &= -\frac{t}{4}\Theta\, D^{-1}(t\Theta)\,\Omega H \Gamma^{MN}\Theta\, L_{MN} W(X,t\Theta) \\ &= -\frac{1}{4t} b^{MN}(t\Theta)\, L_{MN}\, W(X,t\Theta)\,,\end{aligned} \tag{4.33}$$

where

$$b^{MN}(\Theta) = \Theta D^{-1}(\Theta)\,\Omega H \Gamma^{MN}\Theta\,. \tag{4.34}$$

This has a formal solution

$$B(t) = \mathcal{P}\exp\left(-\int_0^t \frac{dt'}{4t'}\, b^{MN}(t'\Theta) L_{MN}\right) B(0)\,, \tag{4.35}$$

so that

$$W(X,\Theta) = B(1) = \mathcal{P}\exp\left(-\int_0^1 \frac{dt}{4t}\, b^{MN}(t\Theta)L_{MN}\right) W_B(X)\,. \tag{4.36}$$

The symbol $\mathcal{P}$ in these expressions denotes that the exponentials are defined by path ordering the operators in the usual manner. Given the quantity $W(X,\Theta)$, the Wilson loop expectation value is given by the integral

$$\langle W\rangle = \int d^{16}\Theta \int \frac{d^6X}{R^4}\delta(X_L^2 - X_T^2 - R^2)W(X,\Theta) \tag{4.37}$$

(recalling the conventions of (3.9)).

At this point it becomes apparent that fermions enter the Wilson loop density in a remarkably simple fashion. The density $W(X,\Theta)$ is simply obtained from the bosonic expression $W_B(X)$ (3.19) by the replacement

$$X^M \longrightarrow \tilde{X}^M = R^M{}_N(\Theta)X^N\,, \tag{4.38}$$

where the matrix $R(\Theta)$ is given by the six-dimensional (fundamental) representation of the operator $\mathcal{P}\exp(-\int dt\, b^{MN}L_{MN}/4t)$ that acts as a rotation on the X coordinates. This observation makes it seem as though the fermionic coordinates can be eliminated from the integrand simply by changing the integration variables from X to $\tilde{X}$. The jacobian for this change of variables is unity so that the fermionic variables disappear from the density and the resulting expression for the Wilson loop expectation value can be written as

$$\langle W\rangle = \int d^{16}\Theta \int \frac{d^6\tilde{X}}{R^4}\delta(\tilde{X}_L^2 - \tilde{X}_T^2 - R^2)W_B(\tilde{X})\,. \tag{4.39}$$

The Grassmann integrals apparently vanish. However, there is an important subtlety due to the fact that the bosonic integral diverges and the expression is really of the form $0\times\infty$. This means that it must be regulated by cutting off the region near the boundary. We will choose to impose the cutoff $X_4 \le \Lambda$, where Λ is large. For fixed $|x|$ this translates into a cutoff $\rho \ge \epsilon$, where $\epsilon = \ell^2/\Lambda$ is small (and we have chosen $\ell = R$), so it is cut off at the boundary of AdS_5. This cutoff is invariant under SO(5) transformations but it is not Poincaré invariant and, as we will discuss in subsection 5.3, does not preserve any of the supersymmetries. In the presence of this cutoff the change of variables (4.38) introduces a dependence on the fermion coordinates in the cutoff-dependent endpoint of the bosonic integral. This suggests a non-zero result may arise as a boundary term. Although it seems probable that the result can be determined by careful analysis of this boundary term we shall proceed by evaluating the integral directly in the original coordinates.

In order to evaluate the integral in (4.37) we will first need to expand the exponential in (4.36) to select the sixteenth power of Θ. This produces a series of powers of L acting on W_B. Eventually only the even powers will survive the bosonic integration and need to be evaluated. A great simplification emerges from observing that

$$b^{MN}(\Theta) = \Theta D^{-1}\Omega H\Gamma^{MN}\Theta = \bar{\Theta}_A\Gamma^{MN}\Theta^A - \bar{\Theta}_A\Theta^B\,\bar{\Theta}_B\Gamma^{MN}\Theta^A\,, \tag{4.40}$$

where we are using the short-hand notation

$$\bar{\Theta}^a_A \equiv H^{ab}\Omega_{AB}\,\Theta^B_b\,. \tag{4.41}$$

All higher powers of Θ in $b^{MN}(\Theta)$ vanish as can be seen by making use of the antisymmetry of $H^{ab}\,\Theta^A_a\Theta^B_b$ under interchange of the Sp(4) indices A and B. An identity that needs to be used here and later on is

$$\begin{aligned}\bar{\Theta}_A\Theta^B\,\bar{\Theta}_B\Gamma^{MN}\Theta^A &= \Omega_{AD}\Omega_{BC}\Theta^A H\Theta^B\,\Theta^C H\Gamma^{MN}\Theta^D \\ &= \frac{1}{3}\varepsilon_{ABCD}\Theta^A H\Theta^B\,\Theta^C H\Gamma^{MN}\Theta^D\,. \end{aligned} \tag{4.42}$$

Antisymmetry on the R-symmetry indices also implies that the only non-zero elements of the terms in (4.40) are those that are in the coset. This means that the only non-zero terms arise when M is a "longitudinal" index of $\mathrm{SO}(2,1)_L$ and N is a "transverse" index of $\mathrm{SO}(3)_T$. Therefore only the subset of L_{MN}'s that are in the coset enter into the exponent of (4.36). It is therefore convenient to decompose the indices under the subgroup H and define

$$\begin{aligned}\Phi &= \bar{\Theta}_A\Gamma^{MN}\Theta^A\,L_{MN} = 2\bar{\Theta}_A\Gamma^{ir}\Theta^A L_{ir}\,, \\ \mathcal{A} &= \bar{\Theta}_A\Theta^B\,\bar{\Theta}_B\Gamma^{MN}\Theta^A L_{MN} = 2\bar{\Theta}_A\Theta^B\,\bar{\Theta}_B\Gamma^{ir}\Theta^A L_{ir}\,, \end{aligned} \tag{4.43}$$

where $i=1,2,3$ label $\mathrm{SO}(2,1)_L$ and $r=4,5,6$ label $\mathrm{SO}(3)_T$. Since the operators Φ and $\mathcal{A}$ do not commute with each other it is essential to take care of the path ordering when expanding the exponential $\mathcal{P}\exp\left(\int du(\Phi-u\mathcal{A})/8\right)$ (where $u=t^2$).

5. Integration over the instanton supermoduli

We will now explicitly evaluate the integral in (4.37) in the presence of a cutoff.

5.1 General properties of the integral

The Grassmann integration selects the terms with sixteen powers of Θ that are obtained by expanding (4.36), which have the schematic form

$$\left(\frac{1}{8!}\Phi^8 - \frac{1}{6!2}\Phi^6\mathcal{A} + \frac{1}{4!2!4}\Phi^4\mathcal{A}^2 - \frac{1}{2!3!8}\Phi^2\mathcal{A}^3 + \frac{1}{4!16}\mathcal{A}^4\right) W_B(X)\,. \tag{5.1}$$

This formula suppresses the combinatorics associated with the path ordering that is non trivial since the operators Φ and $\mathcal{A}$ do not commute. Although the structure of the terms in (5.1) is reminiscent of the expansion of the exponential in (1.1) in powers of $\hat{\varphi}$ and $\hat{A}$, described in the introduction, it is significantly simpler. While (1.1) involved path ordering of matrices in the gauge group that depend on the position around the loop the expression (4.36) does not have this complication. In other words, the use of superconformal symmetries has led to the abelianization of the bosonic connection.

Using the constraint $X_L^2 - X_T^2 = R^2$ (which commutes with the operator $R^M{}_N$ defined in (4.38)) one can think of W_B as a function of $|X_T|$ only, so that

$$L_{ir}W_B(|X_T|) = \frac{X_iX_r}{|X_T|}\frac{\partial}{\partial|X_T|}W_B(|X_T|)\,, \tag{5.2}$$

(recalling that the indices i and r are longitudinal and transverse, respectively). As we will see below the Grassmann integration produces a rather complicated tensor that induces all sorts of contractions of the L's acting on the bosonic invariant $W_B(X)$. We intend to carry out the Grassmann integration first, so we will define

$$F(|X_T|;R) = \int d^{16}\Theta\, W(X,\Theta)\,. \tag{5.3}$$

The general structure of this function can be expressed as

$$\begin{aligned} F(|X_T|;R) &\equiv \int d^{16}\Theta \left[\frac{1}{8!}\Phi^8 + \frac{1}{4!2!4}\Phi^4\mathcal{A}^2 + \frac{1}{4!16}\mathcal{A}^4\right] W_B(|X_T|) \\ &= \sum_{n=1}^{8}\sum_{k=0}^{\left[\frac{n+2}{2}\right]} C^{(n)}_{n+2-2k}|X_T|^{n-2k}R^{2k}\frac{\partial^{(n)}W_B}{\partial|X_T|^n}\,, \end{aligned} \tag{5.4}$$

where the intermediate equation is a symbolic summary of the expansion of the exponential in the integrand. The odd powers of $\mathcal{A}$ in (5.1) have been dropped since they are odd in X and do not contribute to the Wilson loop expectation value since they vanish after integration. A crucial point is that the resulting expression $F(|X_T|;R)$, being H-invariant, can only depend on the single invariant X_T^2.

The integral (4.37) that defines the Wilson loop expectation value has a divergent contribution from the infinite volume of the subspace with constant X_T^2, i.e. from the infinite volume of the integral over each $\mathrm{SO}(3)\times\mathrm{SO}(2,1)$ orbit that comes from the region close to the AdS_5 boundary (near $\rho = 0$ in the original coordinates). We will regularize such divergences by introducing a cutoff $X_4 \le \Lambda = R^2/\epsilon$ which breaks the conformal symmetry. The X_4 integral simply gives (again ignoring a known overall coefficient that will be reinstated at the end)

$$\begin{aligned} \langle W\rangle_\epsilon &= \int^\Lambda \frac{dX_4}{R^4}\int_{X_l^2+X_T^2\le X_4^2-R^2} d^2X_l d^3X_T\,\delta(X_4^2-X_l^2-X_T^2-R^2)\,F(|X_T|;R) \\ &= \frac{1}{R^4}\int_{X_l^2+X_T^2\le\Lambda^2-R^2}\frac{d^2X_l d^3X_T}{\sqrt{X_l^2+X_T^2+R^2}}F(|X_T|;R)\,, \end{aligned} \tag{5.5}$$

where the subscript ϵ indicates the presence of the cutoff $\Lambda = R^2/\epsilon$. Performing the elementary integrals over X_l gives

$$\begin{aligned} \langle W\rangle_\epsilon &= \frac{2\pi}{R^4}\int_{|X_T|\le\sqrt{\Lambda^2-R^2}} d^3X_T\left(\Lambda-\sqrt{X_T^2+R^2}\right)F(|X_T|;R) \\ &= 2\pi\int_{|\mathbf{X}|\le\sqrt{R^2/\epsilon^2-1}} d^3\mathbf{X}\left(\frac{R}{\epsilon}-\sqrt{\mathbf{X}^2+1}\right)F(|\mathbf{X}|;1)\,, \end{aligned} \tag{5.6}$$

where the same rescaling has been used as in (3.21).

The possible divergences of this integral can be analyzed by noting the following properties of $W_B(X)$ and its derivatives, which arise in the definition of $F(|X_T|;R)$ in (5.4). Firstly, note that (5.4) is unaltered if $W_B = -\cos(\pi|X_T|/\sqrt{X_T^2+R^2})$ is replaced by

$W_B^{(0)} \equiv W_B - 1$ which has asymptotic behaviour for large $|X_T|$

$$W_B^{(0)} \equiv W_B - 1 \sim -\frac{\pi^2 R^4}{8|X_T|^4} \,. \tag{5.7}$$

Similarly, derivatives of W_B have asymptotic behaviour

$$W_B^{(n)} = \frac{\partial^{(n)} W_B}{\partial |X_T|^n} \sim (-)^{n+1} \frac{(n+3)!\pi^2}{4!2} \frac{R^4}{|X_T|^{n+4}} \,. \tag{5.8}$$

From these expressions it follows that $\langle W \rangle_\epsilon$ is at most linearly divergent. The fact that the quartic divergence of the purely bosonic integral is absent is a consequence of supersymmetry.

Clearly, a linearly divergent term cannot be present in the exact solution. Such a term, which has the form R/ϵ and is proportional to the circumference of the loop, would represent a breakdown of conformal invariance. However, our calculation introduced a cutoff in the moduli space integration that excludes a region close to the loop. There is therefore a possibility that we have ignored a singular contribution that arises when the instanton touches the loop. Such a term could cancel any apparent singular behaviour in the integral. The coefficient of the term linear in $1/\epsilon$ in (5.6) is finite and so the linear divergence arising from the $\Lambda \to \infty$ limit has the form

$$\frac{2\pi R}{\epsilon} \int d^3 X_T \, F(|X_T|; R) \equiv \frac{2\pi R}{\epsilon} \mathcal{D} \,, \tag{5.9}$$

where $\mathcal{D}$ is a finite coefficient since the integral converges.

The behaviour of the integral can be analyzed in terms of the coefficients $C_m^{(n)}$ in (5.4), noting that

$$\int_0^\infty d|X_T| \, |X_T|^p \, W_B^{(n)} = 0 \,, \tag{5.10}$$

for all $p < n$. After a change of variables one can express the terms with $p = n$ and $p = n+2$ in terms of Bessel functions

$$\begin{aligned} \int d|X_T| \, |X_T|^{n+2} \, W_B^{(n)} &= (-)^{n+1}(n+2)! \int d|X_T| \, |X_T|^2 W_B^{(0)} \\ &= (-)^{n+1} n! \frac{\pi^2}{3} [J_1(\pi) + \pi J_0(\pi)] R \,, \end{aligned} \tag{5.11}$$

$$\int d|X_T| \, |X_T|^n \, W_B^{(n)} = (-)^{n+1} n! \int d|X_T| \, W_B^{(0)} = (-)^{n+1} n! \pi^2 J_1(\pi) R \,. \tag{5.12}$$

$J_1(\pi)$ and $J_0(\pi)$ are incommensurable while the coefficients $C_{n+2}^{(n)}$ are essentially integers. This means that the coefficient of the linear divergence gets independent contributions from these two types of terms. However, the coefficients turn out to satisfy the identity

$$\sum_{n=1}^{8} (-)^n n! C_n^{(n)} = 0 \,, \tag{5.13}$$

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which eliminates the $p = n$ contribution. The remaining contribution is equal to

$$\mathcal{D} \equiv \sum_{n=1}^{8} (-)^{n+1} (n+2)! C_{n+2}^{(n)} . \tag{5.14}$$

The evaluation of $\mathcal{D}$ requires extensive computation that will be described in the next section.

There is also an apparent subleading logarithmic divergence in (5.6). This can be isolated by taking the derivative of $\langle W \rangle_\epsilon$ with respect to ϵ. Since the integrand vanishes at the upper bound one gets

$$\epsilon^2 \frac{d\langle W \rangle_\epsilon}{d\epsilon} \sim \langle W \rangle_{\text{lin}} + \epsilon \, \langle W \rangle_{\text{log}} + \cdots , \tag{5.15}$$

where $\langle W \rangle_{\text{lin}}$ is the coefficient of the linear divergence and $\langle W \rangle_{\text{log}}$ the coefficient of logarithmic divergence. These quantities depend on the polynomials of degree $n+2$ that multiply $W^{(n)}$ in F. A little algebra shows that the coefficients satisfy the condition

$$\langle W \rangle_{\text{log}} = \sum_{n=1}^{8} C_{n+2}^{(n)} (-)^n (n+3)! = 0 , \tag{5.16}$$

so that the logarithmic divergence vanishes.

To summarize, the integral (5.5) gives an expression of the form

$$\langle W \rangle_\epsilon = \mathcal{D} \, \frac{2\pi R}{\epsilon} + \mathcal{F} + O(\epsilon) , \tag{5.17}$$

where $\mathcal{F}$ is the finite integral

$$\mathcal{F} \equiv \langle W \rangle = -\frac{1}{R^4} \int d^3 X_T \, \sqrt{X_T^2 + R^2} \, F(|X_T|; R) . \tag{5.18}$$

Although it is not immediately apparent, the integrand is a total derivative and this integral only gets a contribution from the boundary at $|X_T| = \infty$. This is in line with the expectation based on the original expression for $\langle W \rangle$, which was the integral of a total divergence. Performing the integration in detail and reinstating all the constants that have been dropped up to now gives the expression

$$\langle W \rangle = \frac{g_{\text{YM}}^8}{4! 2^{50} \pi^7} e^{2\pi i \tau} \sum_{n=2}^{8} \sum_{k=2}^{n} (-)^n \frac{(n+3)!}{k+3} C_{n+2}^{(n)} , \tag{5.19}$$

where $\tau = \vartheta/2\pi + 4\pi i/g_{\text{YM}}^2$ is the complexified Yang-Mills coupling.

In order to convert this expression into a number we need to calculate the coefficients $C_n^{(m)}$, which will be the subject of section 5.2.

5.2 Evaluation of the integral

The sixteen-component Grassmann integrations can be performed by decomposing the sixteen-component $SO(4,2) \times SU(4)$ variable Θ_a^A into two eight-component spinors. This is achieved by choosing a basis in which $\Omega_{12} = -\Omega_{21} = 1$ and $\Omega_{34} = -\Omega_{43} = 1$ (with all the remaining components zero). This will allow us to separate Θ_a^A into two $SO(6,2)$ spinors,

$$\hat{\theta} = (\Theta_a^1, \Theta_a^2)\,, \qquad \check{\theta} = (\Theta_a^3, \Theta_a^4)\,. \tag{5.20}$$

The identifications made in these expressions are clarified by considering the decomposition $SO(6,2) \to SO(2,2) \times U(1) \times SU(2)$, which is also a subgroup of $SO(4,2) \times SU(4)$. The $SO(6,2)$ spinors $\hat{\theta}$ and $\check{\theta}$ both transform as $(\mathbf{2},\mathbf{1},\mathbf{2})_+ \oplus (\mathbf{1},\mathbf{2},\mathbf{2})_-$, where the notation refers to the factors in the subgroup (with $\pm$ being the $U(1)$ charge). The overall $SO(4,2) \times SU(4)$ chirality determines the chirality of the $SO(6,2)$ spinors. The R-symmetry indices $(1,2)$ and $(3,4)$ are doublets of the $SU(2)$ factor and the $SO(4,2)$ chirality is the same for all components. Therefore the two $SO(6,2)$ spinors have the same chirality which is inherited from the chiralities of Θ_a^A with respect to the $SO(6)$ R-symmetry and the $SO(4,2)$ conformal symmetry.[10] Recall that this chirality originates from the fact that the BPST instanton is an anti self-dual solution.

From here on we will replace $SO(6,2)$ by $SO(8)$ for notational convenience.[11] The spinor bilinears of relevance to our problem are rewritten in $SO(8)$ notation by using

$$\Theta^1 H \Gamma^{ir} \Theta^2 = \frac{1}{2} \hat{\theta} \gamma^{ir} \hat{\theta}\,, \qquad \Theta^3 H \Gamma^{ir} \Theta^4 = \frac{1}{2} \check{\theta} \gamma^{ir} \check{\theta}\,, \tag{5.21}$$

and

$$\Theta^1 H \Theta^2 = \frac{1}{2} \hat{\theta} \gamma^{78} \hat{\theta}\,, \qquad \Theta^3 H \Theta^4 = \frac{1}{2} \check{\theta} \gamma^{78} \check{\theta}\,, \tag{5.22}$$

where $i = 1,2,3$ are the transverse indices, $r = 4,5,6$ are the longitudinal indices and the $SO(8)$ γ matrices are Clebsch-Gordan coefficients that couple the vector $\mathbf{8_v}$ to the two inequivalent spinors, $\mathbf{8_c}$ and $\mathbf{8_s}$.

With these identifications Φ can be rewritten in the form

$$\Phi = \Omega_{AB} \Theta^A H\, \Gamma^{MN} \Theta^B\, L_{MN} = 2(\hat{\theta} \gamma^{ir} \hat{\theta} + \check{\theta} \gamma^{ir} \check{\theta}) L_{ir}\,. \tag{5.23}$$

After some manipulations the quantity $\mathcal{A}$ defined in (4.43) can be rewritten as

$$\begin{aligned} \mathcal{A} = \frac{8}{3} \Big(& \Theta^1 H \Theta^2\, \Theta^3 \Gamma^{MN} \Theta^4 + \Theta^3 H \Theta^4\, \Theta^1 \Gamma^{MN} \Theta^2 + \\ & + \frac{1}{8} \varepsilon^{PQRSMN}\, \Theta^1 H \Gamma_{PQ} \Theta^2\, \Theta^3 H \Gamma_{RS} \Theta^4 \Big) L_{MN} \\ = \frac{4}{3} & \left(\hat{\theta} \gamma^{ir} \hat{\theta}\, \check{\theta} \gamma^{78} \check{\theta} + \hat{\theta} \gamma^{78} \hat{\theta}\, \check{\theta} \gamma^{ir} \check{\theta} + \frac{1}{2} \varepsilon^{irksmt}\, \hat{\theta} \gamma_{ks} \hat{\theta}\, \check{\theta} \gamma_{mt} \check{\theta} \right) L_{ir}\,. \end{aligned} \tag{5.24}$$

[10]There is plenty of scope for a sign error in determining the absolute sign of the $SO(6,2)$ chirality so we have performed the following calculations allowing for either sign.

[11]This makes no difference to the following discussion and, in any case, we need to make a Wick rotation of one of the time-like coordinates in order to evaluate the instanton contribution.

The signs of the ε terms in these expressions are correlated with the chirality of the instanton solution. This sign changes in the case of an anti instanton. In that case the chirality of both SO(8) spinors also changes, which changes the signs of the other terms in (5.24). Therefore, changing from an instanton to an anti instanton simply reverses the sign of $\mathcal{A}$, which leaves the value of $\langle W \rangle$ unaltered since it only receives contributions from even powers of $\mathcal{A}$.

Substituting (5.23) and (5.24) into (4.36) gives

$$\begin{aligned} W(X,\Theta) &= \mathcal{P}\exp\left\{-\int_0^1 dt' \left(\frac{t'}{2}\Theta\Omega H\Gamma^{ir}\Theta\, L_{ir} + \frac{t'^3}{2}\Theta\Omega H\Theta\, \Theta\Omega H\Gamma^{ir}\Theta\, L_{ir}\right)\right\} W_B(X) \\ &= \mathcal{P}\exp\left\{-\int_0^1 du \left[\frac{1}{4}\left(\hat\theta\gamma^{ir}\hat\theta + \check\theta\gamma^{ir}\check\theta\right) L_{ir} + \right.\right. \\ &\qquad + \frac{u}{6}\left(\hat\theta\gamma^{78}\hat\theta\check\theta\gamma^{ir}\check\theta + \check\theta\gamma^{78}\check\theta\hat\theta\gamma^{ir}\hat\theta + \frac{1}{2}\varepsilon^{jskuir}\hat\theta\gamma_{js}\hat\theta\check\theta\gamma_{ku}\check\theta\right) \times \\ &\qquad \left.\left. \times L_{ir}\right]\right\} W_B(X)\,, \end{aligned} \tag{5.25}$$

where $u = t'^2$. In order to evaluate the Wilson loop expectation value we need to extract the $\hat\theta^8\,\check\theta^8$ term from the expansion of the exponential, taking account of the path ordering, $\mathcal{P}$. The SO(8) Grassmann variables can be integrated out by using the standard result,

$$\int d^8\hat\theta\hat\theta\gamma^{m_1n_1}\hat\theta\cdots\hat\theta\gamma^{m_4n_4}\hat\theta = \hat t_8^{m_1n_1\cdots m_4n_4}\,, \tag{5.26}$$

where $m_r, n_r = 1,\ldots,8$ and $\hat t_8$ is a standard SO(8)-covariant tensor. Similarly, the integral over $\check\theta$ can be expressed in terms of a tensor $\check t_8$. We will be using the explicit form of $\hat t_8$ and $\check t_8$ given in [25] but the range of the indices is restricted to the situation in which $m_r = i_r = 1,2,3,7$ and $n_r = j_r = 4,5,6,8$. Explicitly, the non-vanishing elements of either of these tensors are [25, 26]

$$\begin{aligned} t_8^{i_1r_1i_2r_2i_3r_3i_4r_4} &= \frac{1}{2}\varepsilon^{i_1r_1i_2r_2i_3r_3i_4r_4} - \\ &- \frac{1}{2}\left(\delta^{i_1i_2}\delta^{i_3i_4}\delta^{r_1r_2}\delta^{r_3r_4} + \delta^{i_1i_3}\delta^{i_2i_4}\delta^{r_1r_3}\delta^{r_2r_4} + \delta^{i_1i_4}\delta^{i_2i_3}\delta^{r_1r_4}\delta^{r_2r_3}\right) + \\ &+ \frac{1}{2}\left(\delta^{i_1i_2}\delta^{i_3i_4}\delta^{r_2r_3}\delta^{r_1r_4} + \delta^{i_1i_2}\delta^{i_3i_4}\delta^{r_2r_4}\delta^{r_1r_3} + \delta^{i_1i_4}\delta^{i_2i_3}\delta^{r_2r_4}\delta^{r_1r_3} + \right. \\ &+ \left. \delta^{i_1i_4}\delta^{i_2i_3}\delta^{r_1r_2}\delta^{r_3r_4} + \delta^{i_1i_3}\delta^{i_2i_4}\delta^{r_1r_2}\delta^{r_3r_4} + \delta^{i_1i_3}\delta^{i_2i_4}\delta^{r_1r_4}\delta^{r_2r_3}\right), \end{aligned} \tag{5.27}$$

where the sign of the first term is correlated with the SO(8) chirality of $\hat\theta$ and $\check\theta$. This ε tensor arises from terms that include a γ^{78} factor that can arise from (5.22) and (5.24). The three terms in the first parentheses in (5.27) involve Kronecker deltas that contract the i's in the same sequence as the r's. We will refer to these three terms as "disconnected" contributions. The six terms in the second parentheses in (5.27) give rise to "connected" contributions. A useful shorthand notation is illustrated by considering the contraction

$$\begin{aligned} 2t_8^{i_1r_1i_2r_2i_3r_3i_4r_4} M_{i_1r_1}\cdots M_{i_4r_4} &= -3\,\mathrm{Tr}\,M^2\;\,\mathrm{Tr}\,M^2 + 6\,\mathrm{Tr}\,M^4 \\ &= -3\,\bigcirc\,\bigcirc + 6\,\square\,, \end{aligned} \tag{5.28}$$

where M_{ir} is an arbitrary matrix with left and right indices in separate SO(3)'s. The second line of this equation indicates the contractions diagrammatically. The vertical and horizontal lines indicate contractions of the indices. The coefficients indicate the number of such terms that occur. In the case illustrated above this number is simply an overall multiplicity. However, in the application of interest later on the matrix M_{ir} is replaced by the operator matrix L_{ir}, and it will be important to keep track of the ordering of the indices within each of these different combinations. In other words, many different combinations are subsumed in the notation of (5.28). Furthermore, although the ε term in t_8 did not contribute to (5.28) it does contribute to the expressions that enter into the Wilson loop calculation.

After the Grassmann integration each of the terms in (5.1) gives rise to the sum of a very large number of distinct contractions of powers of L_{ir}. These are evaluated by making repeated use of (5.2) which involves a great deal of computation. Some of the details are discussed in appendix C. In this way the coefficients $C_m^{(n)}$ are calculated and the various contributions to the linear divergence and the finite parts determined. The coefficient, $\mathcal{D}$, of the linear divergence in (5.17) turns out to be a non-zero rational number. For the finite part the result is

$$\langle W \rangle = \mu_{\mathrm{SU}(2)} \frac{\pi^3}{2^{14}\, 3^4\, 5}\,, \tag{5.29}$$

where $\mu_{\mathrm{SU}(2)}$ is the standard measure for a single instanton in $\mathcal{N} = 4$ supersymmetric SU(2) Yang-Mills,

$$\mu_{\mathrm{SU}(2)} = g_{\mathrm{YM}}^8 \frac{1}{2^{34}\pi^{10}}\, e^{2\pi i \tau}\,. \tag{5.30}$$

Since the absolute sign of the chirality of the SO(8) spinors is difficult to determine, we note that with the other choice of chirality the result would be $\mu_{\mathrm{SU}(2)}\, \pi^3 2671/(2^{10}\, 3^4 \cdot 5 \cdot 7)$. The relative simplicity of (5.29) suggests that the first choice is the correct one.

5.3 Cutoff dependence — straight line versus circular loops

We now want to examine whether the result is independent of the cutoff and consistent with supersymmetry. Since the SO(5)-invariant cutoff already breaks all the supersymmetries no supersymmetries survive the introduction of the loop. However, the presence of the loop in the theory with a Poincaré-invariant cutoff leads to further supersymmetry breaking. As discussed in section 1.1 the condition (1.2) (or, equivalently, (4.29)) defines the combinations of supersymmetries that are preserved in the presence of the loop. We argued that the instanton contribution to a circular loop should give a finite value that is independent of the cutoff, but that a straight line should receive a vanishing contribution. In this subsection we shall demonstrate that the expectation value of a circular Wilson loop is independent of the cutoff procedure — more precisely, we will demonstrate that the result does not depend on whether we use the SO(5)-invariant or Poincaré-invariant cutoff, $\rho \geq \epsilon$. In each case the result is given by (5.17). By contrast, we will see that the cutoff $\rho \geq \epsilon$, that is natural for the straight line in $\mathbb{R}^4$, leads to a vanishing finite part, $\mathcal{F}_{\mathrm{line}} = 0$.

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It is easy to see that the SO(5)-invariant cutoff breaks all the supersymmetries, even in the absence of the loop.[12] To see this, recall that a globally defined Killing vector in AdS_5 is timelike, which means that the vector $\epsilon\gamma^\mu\epsilon'$ (where ϵ and ϵ' are supersymmetry parameters, or global Killing spinors) is timelike. However, the surface $X_4 = \Lambda$ preserves SO(5), which is euclidean de Sitter space and has space-like global Killing vectors. Therefore, none of the AdS_5 supersymmetries can be preserved in the cut-off theory.

The Poincaré invariant cutoff $\rho \geq \epsilon$ is adapted to loops on $\mathbb{R}^4$ and is more commonly used in the context of the AdS/CFT correspondence (see references in [1]) and the specific context of holographic renormalization ([27] and references therein). In the absence of the loop this cutoff preserves the sixteen Poincaré supersymmetries, breaking only the sixteen conformal supersymmetries. In terms of the constrained six-dimensional coordinates $\rho = R/(X_4 - X_5)$ (recall that we have set $\ell = R$), so a cutoff at small ρ is equivalent to a cutoff at large light cone coordinate $X^- = X_4 - X_5$. Whereas the endpoint $X_4 = \Lambda = R^2/\epsilon$ intersects the constraint hyperboloid, $(X_4)^2 - (X_5)^2 - (X_0)^2 - (X_1)^2 - (X_2)^2 - (X_3)^2 = R^2$, on a four-sphere, the endpoint $X^- = R^2/\epsilon$ intersects the hyperboloid on a paraboloid.

We will now see that the expressions obtained in earlier sections are in accord with these symmetry considerations, once a divergent perimeter term is subtracted.

5.3.1 Cutoff independence

Whereas the expression (5.6) for a circular loop expectation value was obtained from (5.5) with $X_4 \leq R^2/\epsilon$ we here consider the cutoff $X_4 \leq R^2/\epsilon + X_5$. Integration over the two X_l's followed by the integral over X_4 gives

$$\langle W\rangle'_\epsilon = \frac{1}{R^4}\int d^2X_t\, dX_5 \left(\frac{R^2}{\epsilon} + X_5 - \sqrt{X_T^2 + R^2}\right) F(|X_T|;R)\,, \tag{5.31}$$

where the $'$ indicates the use of the alternative cutoff. This expression is similar to the earlier one (5.6) that used the $X_4 \leq \Lambda$ cutoff, apart from the presence of the term linear in X_5. The boundary conditions require $R^2/\epsilon + X_5 \geq \sqrt{X_T^2 + R^2}$, or

$$X_5 \geq \frac{\epsilon}{2R^2}(X_t^2 + R^2) - \frac{R^2}{2\epsilon}\,. \tag{5.32}$$

This means that there is no upper limit to the X_5 integral, but since (5.31) converges for large X_5 this does not cause a problem. The term linear in X_5 is antisymmetric under $X_5 \to -X_5$ so that it is useful to write it as the sum of two contributions,

$$\int_{-\tilde\Lambda/2}^{\infty} dX_5\, X_5\, F(|X_T|;R) = \int_{-\tilde\Lambda/2}^{\tilde\Lambda/2} dX_5\, X_5\, F(|X_T|;R) + \int_{\tilde\Lambda/2}^{\infty} dX_5\, X_5\, F(|X_T|;R)\,, \tag{5.33}$$

where $\tilde\Lambda = R^2/2\epsilon - \epsilon(X_t^2 + R^2)/2R^2$. The first term in (5.33) vanishes identically. Moreover the second term also vanishes as $1/\tilde\Lambda$ in the $\tilde\Lambda \to \infty$ limit. But $\tilde\Lambda$ is finite only if $X_t^2 \sim R^4/\epsilon^2$, in which case the second term vanishes at least as fast as $|X_t|^{-4}$, so its contribution will vanish after integration over X_t in (5.31). Thus, the X_5 term in (5.31) gives a vanishing contribution.

[12] We are grateful to Gary Gibbons for the following argument.

This means that the expectation value of the circular Wilson loop, $\langle W\rangle'_\epsilon$, has the same value, (5.17), that was found earlier, so the result is not sensitive to the cutoff procedure. We could presumably have regularized the calculation by cutting out any small region of AdS_5 with topology $B^3 \times S^1$ around the loop and obtained the same result.

The presence of a linearly divergent term proportional to the circumference of the loop is obviously inconsistent with conformal invariance and an artifact of the non-supersymmetric cutoff. As argued in the introduction, a more complete treatment would consider the cutoff theory obtained by the spontaneous symmetry breaking $\mathrm{SU}(3) \to \mathrm{SU}(2) \times \mathrm{U}(1)$. In that case there is a dynamical cutoff induced by fluctuations of the W-boson test particle that defines the loop. This radically changes the behaviour of instantons with scales smaller than the inverse W-boson mass, M^{-1}. In the $M \to \infty$ limit such effects are localized on the loop and should generate a perimeter effect. Therefore, in the absence of a complete analysis of such effects, a pragmatic procedure for eliminating the divergent term is to make a local modification of the Wilson loop by absorbing the linearly divergent term into a constant "renormalization" of the test particle mass.

The fact that the terms ignored by our cutoff procedure are localized on the loop suggests that the finite result (5.29) that remains after subtraction of the divergent term is determined uniquely. Since it is independent of the radius the calculation can be repeated for any value of R, including the straight line.[13] In each case the linearly divergent term can be subtracted by the *same* mass "renormalization", with the same finite result.

5.3.2 The straight Wilson line

However, as emphasized in the context of the bosonic model of section 3, there is a different definition of the straight Wilson line in $\mathbb{R}^4$ which utilizes the Poincaré-invariant cutoff. This is pathological from the point of view of S^4 since both the loop and the cutoff surface touch the north pole.

The integrand for the straight line expectation value can be obtained from that of the circular loop (5.6) in the same way as in the bosonic case where (3.26) is obtained from (3.21). The result for the straight line per unit length is simply equal to

$$\langle W(\mathrm{line})\rangle_\epsilon = \frac{1}{\epsilon}\int_{\mathbf{X}\leq\sqrt{R^2/\epsilon^2-1}} d^3\mathbf{X}\, F(|\mathbf{X}|;1)\,. \tag{5.34}$$

This is a pure linear divergence which vanishes after subtracting *the same* mass term that renders the circular loop finite. This is in accord with the constraints of supersymmetry discussed in section 1.1.

6. Discussion and other issues

In summary, after subtracting a linearly divergent term, we have found the finite value in (5.29) for the expectation value of a circular Wilson loop of radius R. The prescription for subtracting the perimeter divergence corresponds to the addition of a counterterm for

[13] In this argument we are considering general radii with $R \neq \ell$.

the (infinite) mass of the test W-boson that defines the loop. We believe that the specific finite result that remains after subtraction of the divergent term is unambiguous since there is no candidate local counterterm that would give a finite contribution. Importantly, the same mass counterterm leads to the vanishing of the instanton contribution to the straight Wilson line, as required by supersymmetry. However, understanding the precise origin of this subtraction is an obvious challenge. It indicates a contribution that is missing from our cutoff prescription and arises from small instantons touching the loop.

The problem is that the BPST instanton solution is not an exact solution of Yang-Mills theory in the presence of the Wilson loop. We argued that the deviation from the exact solution should not be relevant in the semi-classical ($g_{\rm YM} \to 0$) approximation in most of moduli space. However, a zero size instanton touching the loop is a particularly singular configuration that we excluded by our cutoff. No matter how small $g_{\rm YM}$ is, in this region of moduli space the exact solution is very far from the BPST solution. Therefore, the fact that our cutoff neglects the effect of such small-scale instantons localized on the loop might be responsible for the divergent perimeter term. If, instead, the theory were to be regularized by considering the Wilson loop to be the holonomy of a W-boson of large but finite mass, as described in the introduction, these singular small-scale instanton contributions should automatically be incorporated.

The calculations described in this paper raise a number of other interesting issues that we will now turn to.

6.1 Generalization to SU(N)

Having evaluated the instanton contribution to the Wilson loop for gauge group SU(2) we may now consider the extension to SU(N) and other groups. The semi-classical calculation turns out to be very simple. The additional bosonic moduli parameterize the coset $\mathrm{SU}(N)/\,\mathrm{SU}(2)\times\mathrm{SU}(N-2)\times\mathrm{U}(1)$. Since the Wilson loop is gauge invariant the extra $4N-8$ bosonic integrations simply give the volume of the coset. This amounts to multiplying the SU(2) group theoretic coefficient with

$$b_N = \frac{2^{4N-8}\pi^{4N-8}}{(N-1)!(N-2)!}\left(\frac{\rho}{g_{\rm YM}}\right)^{4N-8} . \tag{6.1}$$

Similar considerations apply to other gauge groups. The extra fermionic zero modes are a different story. As is well known, only 16 of them — those associated with the broken supersymmetry and superconformal transformations — are exact. The other $8N-16$, commonly called ν^{Af} and $\bar\nu^B_f$ ($f = 1, \ldots, N-2$) enter into the moduli space action as the quartic interaction

$$S_{4F} = \frac{\pi^2}{4g^2_{\rm YM}\rho^2}\varepsilon_{ABCD}\nu^{Af}\bar\nu^B_f\nu^{Ch}\bar\nu^D_h . \tag{6.2}$$

This means that the Grassmann integration over these variables can be saturated by bringing down $2N-4$ powers of S_{4F} from $e^{-S_{4F}}$. In fact, this is the leading contribution to the Wilson loop expectation value for small $g_{\rm YM}$. Although the fields A and φ in the Wilson loop integrand (1.1) can also soak up the extra fermionic variables such contributions are suppressed by powers of $g_{\rm YM}$. Consequently, the leading contribution can be computed by means of a familiar (Hubbard-Stratonovich) transformation as demonstrated in [28],

$$a_N = \int \prod_{A,f} d\nu_f^A d\bar{\nu}_f^B \left(\frac{g_{\rm YM}^2}{2\pi^2}\right)^{4(N-2)} e^{-S_{4F}} = \frac{(2N-2)!}{2} \left(\frac{g_{\rm YM}^2}{8\pi^2\rho^2}\right)^{2N-4} . \tag{6.3}$$

When combined with the bosonic factor (6.1) this simply modifies the overall coefficient of the measure but does not affect its non-trivial dependence on the collective coordinates and on $g_{\rm YM}$. The result is

$$\langle W \rangle_{{\rm SU}(N)} = \frac{(2N-2)!}{2^{2N-3}(N-1)!(N-2)!} \langle W \rangle_{{\rm SU}(2)} \, . \tag{6.4}$$

The generalization to higher instanton numbers $|k| > 1$ is more difficult for general values of N. First of all it is not possible to use the symmetry arguments that we have used to simplify the problem. On top of that we are faced with the problem of integrating over the unknown multi-instanton moduli space. As shown in [28] drastic simplifications emerge from a saddle point evaluation of the large-N limit, where duality with multi D-instanton effects suggests the multi-instanton moduli space collapses to a copy of AdS_5. This means that the dominant region of moduli space is the one in which all the instantons are at the same position and have the same scale and lie in commuting SU(2) factors inside SU(N) [17, 28, 29]. Once again, in this limit and in the semi-classical approximation, the non-trivial part of the computation is already contained in the SU(2) instanton calculation. The overall measure for arbitrary k at leading order in the $1/N$ expansion given in [28] replaces the factor in (6.4).

6.2 Speculations concerning AdS/CFT

The really interesting question from the point of view of the AdS/CFT correspondence is what happens at strong 't Hooft coupling, $\lambda \equiv g_{\rm YM}^2 N \to \infty$? In this situation it is not possible to neglect the contributions that arise from the ν^{Af} and $\bar{\nu}_f^B$ modes in the exponent of the one-instanton contribution to the Wilson loop density. Roughly speaking each fermion bilinear $\bar{\nu}_f^{[A} \nu^{B]f}$ is replaced by $\sqrt{\lambda}$ while $\bar{\nu}_f^{\{A} \nu^{B\}f}$ is replaced by $g_{\rm YM}$. This gives plenty of scope for reproducing the kind of $\exp\sqrt{\lambda}$ factors that enter into the perturbative expressions. However, for large λ there are many other sources of perturbative corrections to the instanton calculation that also have to be considered, which is a sobering prospect.

Clearly there should be a string theory viewpoint that corresponds to the Yang-Mills calculations even though we are unable to calculate the instanton contribution in the limit of large λ and make a direct comparison with type-IIB supergravity. The arguments of [11] are in accord with the expectation [6, 7] that the Wilson loop in the strongly coupled perturbative sector is determined by the functional integral over all world-sheets bounded by the loop — the world-sheet of minimal area being the dominant configuration. The addition of an instanton corresponds to the addition of a D-instanton in AdS_5. The general supersymmetry considerations of section 1.1 again imply a non-zero contribution to circular loops.

Making quantitative headway with this description does not appear to be simple. However, the strong constraints implied by $\mathrm{SL}(2,\mathbb{Z})$ S-duality might help. This is first seen from the perturbative formula for the Wilson loop of [10] which is in qualitative accord with the supergravity side of the AdS/CFT correspondence where the exponential part of the loop expectation value has the form, $\exp(-A_{\min}) = \exp(\ell^2/\alpha') = \exp\sqrt{g_{\rm YM}^2 N}$. This satisfies the simplest of these conditions

$$\langle W_{1,0}(R;g_s)\rangle \sim \left\langle W_{0,1}\left(R;\frac{1}{g_s}\right)\right\rangle , \tag{6.5}$$

where $g_s = g_{\rm YM}^2/4\pi$ and $W_{p,q}$ is the Wilson loop in which the test charge is a dyon with electric charge p and magnetic charge q. The symbol $\sim$ indicates that the matching is not known to be exact when the prefactors multiplying the exponentials are included. However, an equality is expected by Montonen-Olive duality which equates the expectation value of a Wilson loop for an electrically charged test particle with coupling constant g_s to that of a Wilson loop with a magnetically charged test particle (a 't Hooft loop) with coupling constant $1/g_s$ (when $\vartheta = 0$). This is just the simplest example of the more general statement of how the expectation values should transform under $\mathrm{SL}(2,\mathbb{Z})$. More generally, there is a separate species of Wilson loop for each of the infinite number of different possible dyonic test particles. Under the $\mathrm{SL}(2,\mathbb{Z})$ transformation of the complex coupling,

$$\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d} \tag{6.6}$$

($ad - bc = 1$ with integer a, b, c, d), the expectation value of the Wilson loop must satisfy

$$\langle W_{p,q}(R;\tau)\rangle = \langle W_{r,s}(R;\tau')\rangle , \tag{6.7}$$

where the coprime integers r, s are related to p, q by $\mathrm{SL}(2,\mathbb{Z})$ in the usual manner. This indicates that the 't Hooft loop and all the p, q loops with $q \neq 0$ must have non-trivial dependence on $\tau_1 = \vartheta/2\pi$. In particular, under a shift of τ_1 ($a = 1$, $b = Z$, c=0, $d = 1$) the loop $W_{p,q}$ transforms into $W_{p-Z,q}$. This means that any loop for a test particle carrying a magnetic charge transforms into a different loop under an integer shift of τ_1. The exponential factor $\exp\sqrt{g_{\rm YM}^2 N}$ in the expression for the fundamental Wilson loop has an obvious generalization that has the correct properties, namely

$$\langle W_{p,q}(R;\tau)\rangle \sim P(\tau,\bar\tau)\,\exp\left(|p+q\tau|\sqrt{g_{\rm YM}^2 N}\right) , \tag{6.8}$$

where the prefactor P is undetermined (but was proportional to $\lambda^{-3/4}$ in the limit of large λ in the perturbative sector considered in [10]). Although in the case of the fundamental Wilson loop with $p = 1$, $q = 0$ the exponential factor does not depend on $\tau_1 = \vartheta/2\pi$, it is difficult to imagine that the prefactor has no such dependence. The constraints of $\mathrm{SL}(2,\mathbb{Z})$ covariance typically mix perturbative effects with instanton contributions of the type discussed in this paper.

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6.3 Instanton n-point functions in the presence of a Wilson loop

In addition to $\langle W\rangle$ it is of interest to consider correlation functions of gauge invariant composite operators in the background of the Wilson loop. Such correlation functions encode detailed information about the expansion of the Wilson loop in terms of local operators [13, 30, 31] and are interesting for a variety of other reasons. Among many possible choices of correlation functions, let us focus on two that are particularly special.

Firstly, consider the correlation function $\langle\Lambda(x_1)\cdots\Lambda(x_{16})W\rangle$, where $\Lambda\sim\mathrm{Tr}(F^-_{\mu\nu}\sigma^{\mu\nu}\,\lambda)$ is the composite operator dual to the supergravity dilatino. This correlation function is special because in an instanton background each dilatino contains one factor of ζ. This means that, as in [17], all of the sixteen superconformal collective coordinates have to be absorbed by the dilatini. Consequently, to leading order in $g_{\rm YM}$ the Wilson loop density contains no fermionic coordinates and is given by $W_B(X_T)$. Therefore, the correlation is simply an integral over the bosonic moduli and has the form

$$\langle\Lambda(x_1)\cdots\Lambda(x_{16})W\rangle=\int\frac{d^4x_0d\rho}{\rho^5}\left(\prod_{r=1}^{16}K_{7/2}(x_r;x_0,\rho)\right)\cos\left(\frac{2\pi R^2}{\tilde{\rho}^2+R^2}\right),\tag{6.9}$$

where $\tilde{\rho}$ is defined in (3.13) and (3.14). Although the integral cannot be performed explicitly, the result is manifestly finite and almost certainly non vanishing for generic positions, x_r, of the dilatini (although singularities arise when these operators touch the loop).

The second correlation function we will discuss is $\langle C(x)\,W\rangle$, where the operator $C(x)=\mathrm{Tr}(F^-)^2$ is dual to the complexified dilaton τ and does not absorb any fermionic modes. This means that the final bosonic integral has an additional factor of

$$C(x)=\frac{\rho^4}{[(x-x_0)^2+\rho^2]^4},\tag{6.10}$$

which is simply the classical value of $C(x)$. The calculation can be performed in much the same way as in the case of the pure Wilson loop, leading to

$$\langle C(x)W\rangle=\int\frac{d^4x_0d\rho}{\rho^5}\frac{\rho^4}{[(x-x_0)^2+\rho^2]^4}\int d^8\eta d^8\bar{\xi}\,W[x_0,\rho_0;\eta,\bar{\xi}],\tag{6.11}$$

where $W[x_0,\rho_0;\eta,\bar{\xi}]$ is the Wilson loop density derived earlier. The integration over the fermionic moduli can be performed exactly as in the case of the loop with no additional insertions. Further integration over the insertion point x gives the Ward identity

$$\int d^4x\langle C(x)W\rangle=\frac{\partial}{\partial\tau}\langle W\rangle\,.\tag{6.12}$$

Similarly,

$$\int d^4x\langle\bar{C}(x)W\rangle=\frac{\partial}{\partial\bar{\tau}}\langle W\rangle\,.\tag{6.13}$$

The sum of (6.12) and (6.13) gives an identity for $\partial\langle W\rangle/\partial\vartheta_{\rm YM}$. In sectors with non-zero instanton number the left-hand sides of (6.12) and (6.13) appear to be very different since $\bar{C}$ contains eight fermionic moduli (since $\bar{C}\sim(\mathrm{Tr}\,F^+)^2$) whereas C contains none. This indicates that $\partial\langle W\rangle/\partial\vartheta_{\rm YM}\neq 0$, which would mean that $\langle W\rangle$ has a non-trivial dependence on $\vartheta_{\rm YM}$.

Other correlation functions of composite gauge-invariant operators in a Wilson loop background can be computed in a similar manner, such as those involving the lowest chiral primary operator Q^{ij}. These kinds of calculations may reveal interesting information concerning the operator product expansion of the Wilson loop [13, 31] and the structure of non-local operators, much as the operator product expansion of correlation functions [18, 32, 33, 34, 35] has revealed a rich structure of local scaling operators.

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A. Six-dimensional representation of conformal transformations of AdS_5

In this appendix we will describe the finite SO(2,2) transformations that map (x^μ, ρ) into $(x^{\mu\prime}, \rho')$, where both points have the same value of the SO(2,2) invariant, U. We will use a 6×6 matrix representation of the elements of the group. In this representation the generators of translations and special conformal transformations are given by

$$(P_\mu)^M{}_N = \delta^M_\mu \eta_{N4} + \delta^M_\mu \eta_{N5} + \eta_{\mu N} \delta^M_4 + \eta_{\mu N} \delta^M_5 \tag{A.1}$$

and

$$(K_\mu)^M{}_N = \delta^M_\mu \eta_{N4} - \delta^M_\mu \eta_{N5} + \eta_{\mu N} \delta^M_4 - \eta_{\mu N} \delta^M_5 \,. \tag{A.2}$$

The cube of any of these matrices vanishes, $P^3_\mu = K^3_\mu = 0$, which makes it very easy to determine the finite transformations. A finite translation is represented by

$$(e^{a^\mu P_\mu})^M{}_N X^N = X^M + (a^\mu P_\mu)^M{}_N X^N + \frac{1}{2}(a^\mu a^\nu P_\mu P_\nu)^M{}_N X^N \,, \tag{A.3}$$

which implies

$$x^{\mu\prime} = x^\mu + a^\mu \,, \qquad \rho' = \rho \,. \tag{A.4}$$

Similarly, a special conformal transformation is represented by

$$(e^{b^\mu K_\mu})^M{}_N X^N = X^M + (b^\mu K_\mu)^M{}_N X^N + \frac{1}{2}(b^\mu b^\nu K_\mu K_\nu)^M{}_N X^N \,, \tag{A.5}$$

which implies

$$x^{\mu\prime} = \frac{x^\mu + b^\mu (x^2 + \rho^2)}{1 + 2b \cdot x + b^2(\rho^2 - x^2)} \,, \qquad \rho' = \frac{\rho}{1 + 2b \cdot x + b^2(\rho^2 + x^2)} \,. \tag{A.6}$$

It is important that these transformations preserve the boundary $\rho = 0$ and that they reduce to the usual four-dimensional conformal transformations on the boundary.

We are interested in the transformations generated by

$$\Pi_l^+ = R\,P_l + \frac{1}{R}K_l\,, \qquad \Pi_t^- = R\,P_t - \frac{1}{R}K_t\,. \tag{A.7}$$

Using the matrix representation one finds

$$e^{\alpha\Pi^+} = 1 + \frac{1}{2}\Pi^+\sinh 2\alpha + \frac{1}{4}(\Pi^+)^2(\cosh 2\alpha - 1)\,. \tag{A.8}$$

The action on the coordinates is easiest to describe by decomposing x^μ into components parallel $x_\parallel$ to and orthogonal $x_\perp$ to the vector α^μ, which gives

$$\begin{aligned}
x_\parallel'^{(+)} &= \frac{x_\parallel + a\left(1+\frac{x^2+\rho^2}{R^2}\right)}{\frac{1}{2}\left(1+\sqrt{1-\frac{4a^2}{R^2}}\right)+2\frac{ax_\parallel}{R^2}+\frac{1}{2R^2}\left(1-\sqrt{1-\frac{4a^2}{R^2}}\right)(\rho^2-x^2)}\,,\\
x_\perp'^{(+)} &= \frac{\sqrt{1-\frac{4a^2}{R^2}}x_\perp}{\frac{1}{2}\left(1+\sqrt{1-\frac{4a^2}{R^2}}\right)+2\frac{ax_\parallel}{R^2}+\frac{1}{2R^2}\left(1-\sqrt{1-\frac{4a^2}{R^2}}\right)(\rho^2-x^2)}\,,\\
\rho'^{(+)} &= \frac{\sqrt{1-\frac{4a^2}{R^2}}\rho}{\frac{1}{2}\left(1+\sqrt{1-\frac{4a^2}{R^2}}\right)+2\frac{ax_\parallel}{R^2}+\frac{1}{2R^2}\left(1-\sqrt{1-\frac{4a^2}{R^2}}\right)(\rho^2-x^2)}\,,
\end{aligned} \tag{A.9}$$

where

$$a = \frac{R}{2}\tanh 2\alpha\,. \tag{A.10}$$

Similarly using the matrix representation one finds

$$e^{\beta\Pi^-} = 1 + \frac{1}{2}\Pi^-\sin 2\beta - \frac{1}{4}(\Pi^-)^2(\cos 2\beta - 1)\,. \tag{A.11}$$

The transformation of the coordinates, now decomposed into components perpendicular and parallel to the vector β^μ, is given by

$$\begin{aligned}
x_\parallel'^{(-)} &= \frac{x_\parallel + b\left(1-\frac{x^2+\rho^2}{R^2}\right)}{\frac{1}{2}\left(1+\sqrt{1+\frac{4b^2}{R^2}}\right)-2\frac{bx_\parallel}{R^2}+\frac{1}{2R^2}\left(\sqrt{1+\frac{4b^2}{R^2}}-1\right)(\rho^2-x^2)}\,,\\
x_\perp'^{(-)} &= \frac{\sqrt{1+\frac{4b^2}{R^2}}x_\perp}{\frac{1}{2}\left(1+\sqrt{1+\frac{4b^2}{R^2}}\right)-2\frac{bx_\parallel}{R^2}+\frac{1}{2R^2}\left(\sqrt{1+\frac{4b^2}{R^2}}-1\right)(\rho^2-x^2)}\,,\\
\rho'^{(-)} &= \frac{\sqrt{1+\frac{4b^2}{R^2}}\rho}{\frac{1}{2}\left(1+\sqrt{1+\frac{4b^2}{R^2}}\right)-2\frac{bx_\parallel}{R^2}+\frac{1}{2R^2}\left(\sqrt{1+\frac{4b^2}{R^2}}-1\right)(\rho^2-x^2)}\,,
\end{aligned} \tag{A.12}$$

where

$$b = \frac{R}{2}\tan 2\beta\,. \tag{A.13}$$

Since $\Pi_l^{(+)}$ and $\Pi_t^{(-)}$ commute with one another, one can easily combine the two transformations and explicitly determine the parameters $a^l = a_0^l$ and $b^t = b_0^t$ that give the transformation taking the instanton to the centre of the loop $x_0' = 0$. The scale $\tilde{\rho}$ is thereby determined.

B. Killing supervectors of the instanton superspace

In this appendix we shall present the detailed expressions for the coefficients Σ^A that enter into (4.15) and (4.19). We will consider the dependence on the different constants — $x_\oplus$, $\lambda_\oplus$, $w_\oplus$, $b_\oplus$, $\eta_\oplus$, $\bar{\eta}_\oplus$, $\bar{\xi}_\oplus$ and $\xi_\oplus$ — in turn.

1) Translations, $x_\oplus^\mu$:

$$\Sigma(x_\oplus) = x_\oplus \left(e^{-\lambda} P + e^{-\lambda/2} \bar{\xi} \sigma Q\right). \tag{B.1}$$

2) Dilations, $\lambda_\oplus$:

$$\Sigma(\lambda_\oplus) = \lambda_\oplus \left(D - \frac{1}{2} e^{+\lambda/2} \bar{\xi} \bar{S} + \frac{1}{2} e^{-\lambda/2} (\eta + 2\bar{\xi} x \cdot \sigma) Q + e^{-\lambda} x \cdot P\right). \tag{B.2}$$

3) Rotations, $\omega_\oplus^{\mu\nu}$:

$$\begin{aligned}\Sigma(\omega_\oplus) = \frac{1}{2}\omega_\oplus^{\mu\nu} \Big(& J_{\mu\nu} + e^{-\lambda}[x_\mu P_\nu - x_\nu P_\mu] - \frac{1}{2} e^{+\lambda/2} \bar{\xi} \bar{\sigma}_{\mu\nu} \bar{S} - \\ & - \frac{1}{2} e^{-\lambda/2} \eta \sigma_{\mu\nu} Q + e^{-\lambda/2} \bar{\xi} \bar{\sigma}_\mu x_\nu Q \Big).\end{aligned} \tag{B.3}$$

4) Conformal boosts, $b_\oplus^\mu$:

$$\begin{aligned}\Sigma(b_\oplus) = b_\oplus^\mu \Big(& e^{+\lambda} K_\mu - x_\mu D - \frac{1}{2} x^\nu J_{\nu\mu} e^{-\lambda} \left[x_\mu x P - \frac{1}{2} x^2 P_\mu\right] + \\ & + e^{-\lambda/2} \bar{\xi} \left[x_\mu x \sigma - \frac{1}{2} x^2 \sigma_\mu\right] Q + e^{+\lambda/2} \left[\eta \sigma_\mu \bar{S} + \frac{1}{2} x_\mu \bar{\xi} \bar{S} - \frac{1}{2} x^\nu \bar{\xi} \bar{\sigma}_{\nu\mu} \bar{S}\right] - \\ & - \frac{1}{2} e^{-\lambda/2} \left[x_\mu \eta Q + x^\nu \eta \sigma_{\nu\mu} Q\right] \Big).\end{aligned} \tag{B.4}$$

5) Left supersymmetries, $\eta_\oplus^{\alpha A}$:

$$\Sigma(\eta_\oplus) = e^{-\lambda/2} \eta_\oplus Q. \tag{B.5}$$

6) Right supersymmetries, $\bar{\eta}^{\dot{\alpha}}_{\oplus A}$:

$$\begin{aligned}\Sigma(\bar{\eta}_\oplus) = \bar{\eta}_\oplus \Big(& e^{-\lambda/2} \bar{Q} + e^{-\lambda/2} \bar{\sigma} \eta \bar{\xi} \bar{\sigma} Q + e^{-\lambda} \bar{\sigma}^\mu \eta P - \bar{\xi} D - \frac{1}{4} \bar{\sigma}^{\mu\nu} \bar{\xi} J_{\mu\nu} - \frac{1}{2} \hat{\Gamma}^{ij}\, \bar{\xi}\, T_{ij} - \\ & - \frac{1}{4} \bar{\xi} e^{+\lambda/2} \bar{\xi} \bar{S} - \frac{1}{16} \bar{\sigma}^{\mu\nu} \bar{\xi} e^{+\lambda/2} \bar{\xi} \bar{\sigma}_{\mu\nu} \bar{S} + \frac{1}{8} \hat{\Gamma}_{ij} \bar{\xi} e^{+\lambda/2} \bar{\xi} \hat{\Gamma}^{ij} \bar{S} \Big).\end{aligned} \tag{B.6}$$

7) Right conformal supersymmetries, $\bar{\xi}_\oplus^{\dot{\alpha} A}$:

$$\Sigma(\bar{\xi}_\oplus) = \bar{\xi}_\oplus (e^{+\lambda/2} \bar{S} + e^{-\lambda/2} x \cdot \sigma Q). \tag{B.7}$$

8) Left conformal supersymmetries, $\xi^{\alpha}_{\oplus A}$:

$$\begin{aligned}
\Sigma(\xi_\oplus) = \xi_\oplus \Big(& e^{+\lambda/2}S + e^{+\lambda}\sigma\bar{\xi}K + x\cdot\sigma\bar{\sigma}\eta\left[e^{-\lambda}P + e^{-\lambda/2}\bar{\xi}\bar{\sigma}Q\right] + \\
& + \eta\left[D + \frac{1}{4}e^{-\lambda/2}\eta Q - \frac{1}{4}e^{+\lambda/2}\bar{\xi}\bar{S}\right] + \frac{1}{4}\sigma^{\mu\nu}\eta\left[J_{\mu\nu} + \frac{1}{4}e^{-\lambda/2}\eta\sigma_{\mu\nu}Q\right] + \\
& + \frac{1}{2}\hat{\Gamma}^{ij}\eta\left[T_{ij} + \frac{1}{4}e^{-\lambda/2}\eta\hat{\Gamma}_{ij}Q + \frac{1}{4}e^{+\lambda/2}\bar{\xi}\hat{\Gamma}_{ij}\bar{S}\right] + x\cdot\sigma\bar{\xi}\left[D - \frac{1}{4}e^{+\lambda/2}\bar{\xi}\bar{S}\right] + \\
& + \frac{1}{4}x\cdot\sigma\bar{\sigma}^{\mu\nu}\bar{\xi}\left[J_{\mu\nu} + \frac{1}{4}e^{+\lambda/2}\bar{\xi}\bar{\sigma}_{\mu\nu}\bar{S}\right] + \\
& + \frac{1}{2}x\cdot\sigma\hat{\Gamma}^{ij}\bar{\xi}\left[T_{ij} + \frac{1}{4}e^{+\lambda/2}\bar{\xi}\hat{\Gamma}_{ij}\bar{S}\right]\Big) . \qquad \text{(B.8)}
\end{aligned}$$

Using these expressions in (4.21) and the expressions for the inverse supervielbeins (4.12) we can deduce the superisometries, as follows.

(a) Left supersymmetry:

$$\delta_Q x^\mu = 0\,, \qquad \delta_Q\lambda = 0\,, \qquad \delta_Q\eta^{\alpha A} = \eta^{\alpha A}_{\oplus}\,, \qquad \delta_Q\bar{\xi}^{\dot{\alpha}A} = 0\,. \qquad \text{(B.9)}$$

(b) Right conformal supersymmetry:

$$\delta_{\bar{S}} x^\mu = 0\,, \qquad \delta_{\bar{S}}\lambda = 0\,, \qquad \delta_{\bar{S}}\eta^{\alpha A} = \bar{\xi}^{A}_{\oplus\dot{\alpha}}\bar{\sigma}^{\dot{\alpha}\alpha}_{\mu}x^\mu\,, \qquad \delta_{\bar{S}}\bar{\xi}^{\dot{\alpha}A} = \bar{\xi}^{\dot{\alpha}A}_{\oplus}\,. \qquad \text{(B.10)}$$

(c) Right supersymmetry:

$$\begin{aligned}
\delta_{\bar{Q}} x^\mu &= \bar{\eta}_\oplus\sigma^\mu\eta\,, \qquad \delta_{\bar{Q}}\lambda = -\bar{\eta}_\oplus\bar{\xi}\,, \qquad \delta_{\bar{Q}}\eta^{\alpha A} = 0\,, \\
\delta_{\bar{Q}}\bar{\xi}^{\dot{\alpha}A} &= -\frac{1}{4}\bar{\eta}_\oplus\bar{\xi}\,\bar{\xi}^{\dot{\alpha}A} + \frac{1}{16}\bar{\eta}_\oplus\bar{\sigma}^{\mu\nu}\bar{\xi}\,(\bar{\xi}\bar{\sigma}_{\mu\nu})^{\dot{\alpha}A} + \frac{1}{16}\bar{\eta}_\oplus\hat{\Gamma}^{ij}\bar{\xi}\,(\bar{\xi}\hat{\Gamma}_{ij})^{\dot{\alpha}A}\,. \qquad \text{(B.11)}
\end{aligned}$$

(d) Left conformal supersymmetry:

$$\begin{aligned}
\delta_S x^\mu &= \xi_\oplus\sigma\cdot x\bar{\sigma}^\mu\eta + \frac{1}{2}e^{2\lambda}\xi_\oplus\sigma^\mu\bar{\xi} + \xi_\oplus\sigma_\nu\bar{\xi}\,(\eta^{\mu\nu}x^2 - 2x^\mu x^\nu)\,, \\
\delta_S\lambda &= -\xi_\oplus(\eta + \sigma\cdot x\bar{\xi})\,, \\
\delta_S\eta^{\alpha A} &= -\frac{1}{4}\xi_\oplus\eta\eta^{\alpha A} + \frac{1}{16}\xi_\oplus\sigma^{\mu\nu}\eta\,(\eta\sigma_{\mu\nu})^{\alpha A} + \frac{1}{16}\xi_\oplus\hat{\Gamma}^{ij}\eta\,(\eta\hat{\Gamma}_{ij})^{\alpha A}\,, \\
\delta_S\bar{\xi}^{\dot{\alpha}A} &= -\frac{1}{4}\xi_\oplus(\eta + \sigma\cdot x\bar{\xi})\bar{\xi}^{\dot{\alpha}A} + \frac{1}{16}\xi_\oplus\hat{\Gamma}^{ij}(\eta + \sigma\cdot x\bar{\xi})\,(\bar{\xi}\hat{\Gamma}_{ij})^{\dot{\alpha}A}\,. \qquad \text{(B.12)}
\end{aligned}$$

From these symmetry transformations it is simple to deduce the OSp$(2,2|4)$ transformations generated by (here we are setting $R = 1$ for simplicity of notation)

$$G_A = \sigma^{12}Q_A + \Omega_{AB}S^B\,, \qquad \bar{G}^A = \bar{\sigma}^{12}\bar{Q}^A + \Omega^{AB}\bar{S}_B\,, \qquad \text{(B.13)}$$

with constant parameters that we denote by $\epsilon^A_\oplus, \bar{\epsilon}_{\oplus A}$. These are

$$\begin{aligned}
\delta_G x^\mu &= \Omega_{AB}\epsilon^A_\oplus(\sigma^\mu\bar{\xi}^B e^{2\lambda} + \sigma\cdot x\bar{\sigma}^\mu\eta^B)\,, \\
\delta_G\lambda &= \Omega_{AB}\epsilon^A_\oplus(\eta^B + \sigma\cdot x\bar{\xi}^B)\,, \\
\delta_G\eta^A &= \sigma^{12}\epsilon^A_\oplus - \frac{1}{4}\Omega_{BC}\epsilon^B_\oplus\left(\eta^C\eta^A + \frac{1}{4}\sigma^{\mu\nu}\eta^C(\eta\sigma_{\mu\nu})^A + \hat{\Gamma}^{C}_{ij\;D}\eta^D(\eta\hat{\Gamma}^{ij})^A\right), \\
\delta_G\bar{\xi}^A &= -\frac{1}{4}\Omega_{BC}\epsilon^B_\oplus\left((\eta^C + \sigma\cdot x\bar{\xi}^C)\bar{\xi}^A + 2\hat{\Gamma}^{ij\;C}_{D}(\eta^D + \sigma\cdot x\bar{\xi}^D)(\bar{\xi}\hat{\Gamma}_{ij})^A\right), \qquad \text{(B.14)}
\end{aligned}$$

and

$$\begin{aligned}
\delta_{\bar G} x^\mu &= \bar\epsilon_{\oplus A}\bar\sigma^{12}\sigma^\mu \eta^A\,,\\
\delta_{\bar G}\lambda &= \bar\epsilon_{\oplus A}\bar\sigma^{12}\bar\xi^A\,,\\
\delta_{\bar G}\eta^A &= \Omega^{AB}\bar\epsilon_{\oplus B}\bar\sigma\cdot x\,,\\
\delta_{\bar G}\bar\xi^A &= \Omega^{AB}\bar\epsilon_{\oplus B} - \frac{1}{4}\bar\epsilon_{\oplus B}\bar\sigma^{12}\left(\bar\xi^B\bar\xi^A + \frac{1}{4}\bar\sigma^{\mu\nu}\bar\xi^B(\bar\xi^A\bar\sigma^{\mu\nu}) + \frac{1}{4}\hat\Gamma^B_{ij\ C}\bar\xi^C\,(\bar\xi\hat\Gamma^{ij})^A\right).
\end{aligned} \tag{B.15}$$

C. Computation of the integral

The various terms that need to be considered in the integrand of the Wilson loop are those denoted by Φ^8, $\Phi^4\mathcal{A}^2$ and $\mathcal{A}^4$ in (5.4). In this appendix we will sketch the systematics of the calculation of each of these terms and the way in which they contribute to the linearly divergent part of the integral as well as to the finite part. The precise details are too complicated to warrant presentation here (but can be obtained by direct communication with the authors). We will discuss only the case corresponding to the choice of chirality for the SO(8) spinors that leads to the simpler result of equation (5.29).

C.1 The Φ^8 terms

Since there is no issue of noncommutativity for these terms they are relatively straightforward to evaluate. The expansion of eight powers of $(\hat\theta\gamma^{ir}\hat\theta + \check\theta\gamma^{ir}\check\theta)$ gives

$$\Phi^8|_{8\times 8} = \sum_{(70)} \hat\theta\gamma^{i_1r_1}\hat\theta\cdots\hat\theta\gamma^{i_4r_4}\hat\theta\check\theta\gamma^{i_5r_5}\check\theta\ldots\check\theta\gamma^{i_8r_8}\check\theta L_{i_1r_1}\cdots L_{i_8r_8}\,, \tag{C.1}$$

where the subscript (70) indicates a sum over the $8!/4!4! = 70$ terms that contain four pairs of $\hat\theta$'s and four pairs of $\check\theta$'s. Using (5.26) in order to perform the integrals over $\hat\theta$ and $\check\theta$ gives a contribution to the Wilson loop of the form

$$\begin{aligned}
\Phi^8 &= \sum_{(70)} \hat{t_8}^{i_1r_1\cdots i_4r_4} t_8{}^{i_5r_5\cdots i_8r_8} L_{i_1r_1}\cdots L_{i_8r_8}\\
&= \sum_{(70)} ((6)\delta^{1234} - (3)\delta^{12}\delta^{34})((6)\delta^{5678} - (3)\delta^{56}\delta^{78})L_{(1)}\cdots L_{(8)}\,.
\end{aligned} \tag{C.2}$$

In this expression the coefficients (6) and (3) indicate the six connected contributions and the three disconnected contributions that enter into each of the t_8's. We have also indicated the index contractions impressionistically. Each of the terms that arises from (C.2) has the form of a specific contraction between the eight powers of L_{ir}. Because each term in the sum over permutations factorizes into the product of two t_8's the contractions between the L's also factorize into two groups — the product of contractions on the indices $i_1r_1\cdots i_4r_4$ and the contractions on $i_5r_5\cdots i_8r_8$. Performing these contractions gives a total contribution from the Φ^8 term of the form

$$\begin{aligned}
\frac{1}{8!}\Phi^8\,W_B(X) &= \frac{1}{8!}70\times\left\{36\ \Box\ \Box - 36\ \Box\ \bigcirc\ \bigcirc + 9\ \bigcirc\ \bigcirc\ \bigcirc\ \bigcirc\right\}\\
&= \frac{1}{8!}\,(L44 - L422 + L2222)\,.
\end{aligned} \tag{C.3}$$

The notation $L44$ indicates the collection of all terms that are connected on the $i_1, r_1 \cdots i_4$, r_4 indices as well as on the $i_5, r_5 \cdots i_8, l_8$ indices. Since the individual L_{ir}'s do not commute with each other it is important to keep the correct ordering. As a result there are $70 \times 36 = 2520$ distinct terms in $L44$, corresponding to the different possible contractions. The terms in $L422$ are those that are connected on the $i_1, r_1 \cdots i_4, r_4$ indices but disconnected on $i_5, r_5 \cdots i_8, r_8$, as well as those that are related by interchanging $1,2,3,4$ with $5,6,7,8$. This also has 2520 terms. Likewise $L2222$ indicates those terms that are disconnected in both sets of indices. There are $70 \times 9 = 630$ such terms.

Each of the quantities $L44$, $L422$ and $L2222$ has the form of a polynomial in $|X_T|$ multiplying a differential of $W_B(X)$, as indicated in (5.4). Since each of the many thousands of terms involves the product of eight L_{ir}'s there is a substantial computational problem, for which we have made extensive use of REDUCE in order to determine the explicit expressions. Substituting these in (5.6) we can extract the Φ^8 contribution to the linear divergence,

$$\mathcal{D}1 = \frac{1}{8!}\int_0^\infty d|X_T|\,|X_T|^2\,\Phi^8\,W_B(X) = \frac{1}{8!}\,(L44 - L422 + L2222) = \frac{35}{192}\,. \tag{C.4}$$

Similarly the Φ^8 contribution to the finite term is given by

$$\mathcal{F}1 = \frac{1}{8!}\int_0^\infty d|X_T|\,|X_T|^2\,\sqrt{X_T^2 + R^2}\,\Phi^8\,W_B(X) = -\frac{1}{45}\,. \tag{C.5}$$

C.2 The $\Phi^4\mathcal{A}^2$ terms

The terms $\Phi^4\mathcal{A}^2$ need special attention since Φ and $\mathcal{A}$ do not commute. There are 15 distinct orderings of the Φ's and $\mathcal{A}$'s. In the path ordered expansion of the exponential in (4.36) there is a factor that arises from the u integrations that depends on which of these fifteen orderings is being considered. Thus, if the two $\mathcal{A}$ operators are the pth and qth positions in the chain of six operators this factor is proportional to

$$a_{pq} = \int_0^1 du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_5} du_6\, u_p\, u_q\,, \tag{C.6}$$

where $u_r = t_r^2$.

We will illustrate the procedure for the simple example in which the two $\mathcal{A}$'s are the last operators in the chain. This has the form

$$\begin{aligned}&\left(\frac{4}{3}\right)^2 \Big((\hat\theta\gamma^{pq}\hat\theta + \check\theta\gamma^{pq}\check\theta)L_{pq}\Big)^4 \times \\ &\quad\times \Big((2\hat\theta\gamma^{78}\hat\theta\check\theta\gamma^{ir}\check\theta + 2\hat\theta\gamma^{ir}\hat\theta\check\theta\gamma^{78}\check\theta + \varepsilon^{irksmt}\hat\theta\gamma_{ks}\hat\theta\check\theta\gamma_{mt}\check\theta)L_{ir}\Big)^2\,.\end{aligned} \tag{C.7}$$

The Grassmann integration selects the terms with eight $\hat\theta$'s and eight $\check\theta$'s in the expansion of this expression. It is convenient to group these terms according to the number of ε

tensors they contain.[14] The term with no ε tensors is

$$\left(\frac{8}{3}\right)^2 \left\{ \sum_{(6)} \hat\theta\gamma^{i_1r_1}\hat\theta\hat\theta\gamma^{i_2r_2}\hat\theta(\hat\theta\gamma^{78}\hat\theta)^2\check\theta\gamma^{i_3r_3}\check\theta\cdots\check\theta\gamma^{i_6r_6}\check\theta + (\hat\theta \leftrightarrow \check\theta) + \right. \tag{C.8}$$
$$\left. + \sum_{(6)} \hat\theta\gamma^{i_1r_1}\hat\theta\hat\theta\gamma^{i_2r_2}\hat\theta\hat\theta\gamma^{i_3r_3}\hat\theta\hat\theta\gamma^{78}\hat\theta\check\theta\gamma^{i_4r_4}\check\theta\check\theta\gamma^{i_5r_5}\check\theta\check\theta\gamma^{i_6r_6}\check\theta\check\theta\gamma^{78}\check\theta \right\} L_{i_1r_1}\cdots L_{i_6r_6}\,,$$

where the sums are over the $6 = 4!/2!2!$ distributions of 4 elements into two groups of 2. Integrating over $\hat\theta$ and $\check\theta$ gives

$$\left(\frac{8}{3}\right)^2 \sum_{(6)} \left(t^{i_1r_1i_2r_27878}t^{i_3r_3\cdots i_6r_6} + t^{i_1r_1\cdots i_4r_4}t^{i_5r_5i_6r_67878} + 2t^{i_1r_1.i_3r_378}t^{i_4r_4.i_6r_678}\right) \times$$
$$\times L_{i_1r_1}\cdots L_{i_6r_6} = \left(\frac{8}{3}\right)^2 \big[6(-\delta^{12})((6)\delta^{3456} - (3)\delta^{34}\delta^{56}) +$$
$$+ 6((6)\delta^{1234} - (3)\delta^{12}\delta^{34})(-\delta^{56}) + 12((6)\delta^{12}\delta^{34}\delta^{56} +$$
$$+(12)\delta^{123456} - (18)\delta^{1234}\delta^{56})\big]\, L_{(1)}\cdots L_{(6)}\,. \tag{C.9}$$

As before, the notation is symbolic, the numbers in parentheses indicating the number of distinct permutations involved. The expression (C.9) is again evaluated by a REDUCE programme. The terms with one or two ε's in the expansion of (C.7) must also be included by a similar analysis in order to complete the first of the fifteen permutations. The result is a contribution to the $\Phi^4\mathcal{A}^2$ term that is a sum of terms with the structures $L6$, $L42$ and $L222$ — in the earlier terminology these are connected, partially connected and disconnected, respectively.

A similar procedure is carried out for each of the other fourteen distinct orderings of the Φ and $\mathcal{A}$ operators in the chain. They each give rise to contributions to the $\Phi^4\mathcal{A}^2$ term that have the structures $L6$, $L42$ and $L222$. The result of the REDUCE computation of the sum of these terms gives the contribution to the linearly divergent part, (5.6),

$$\mathcal{D}2 = \frac{1}{4!2!}\int_0^\infty d|X_T|\,|X_T|^2\,\Phi^4\mathcal{A}^2\,W_B(X) = -\frac{7}{10}\,. \tag{C.10}$$

The contribution to the finite part is

$$\mathcal{F}2 = \frac{1}{4!2!}\int_0^\infty d|X_T|\,|X_T|^2\,\sqrt{X_T^2+R^2}\,\Phi^4\mathcal{A}^2\,W_B(X) = \frac{11}{5}\,. \tag{C.11}$$

C.3 The $\mathcal{A}^4$ terms

The last set of terms that arises in the expansion of the integrand are those of the form $\mathcal{A}^4$ terms for which there is again no problem with noncommutativity.

[14] We are here referring to the number of explicit ε's in the expansion of (C.7). Other factors of ε arise from the definition of the t_8 tensors.

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It is again convenient to group terms according to the number of ε tensors. The terms with no ε come from the expansion of

$$\Big((\hat\theta\gamma^{ir}\hat\theta\check\theta\gamma^{78}\check\theta + \hat\theta\gamma^{ir}\hat\theta\check\theta\gamma^{ir}\check\theta)_{ir}\Big)^4 = \sum_{(70)} \hat\theta\gamma^{i_1r_1}\hat\theta\cdots\hat\theta\gamma^{i_4r_4}\hat\theta\check\theta\gamma^{i_5r_5}\check\theta\cdots\check\theta\gamma^{i_8r_8}\check\theta \times \\ \times\, L_{i_1r_1}\cdots L_{i_8r_8}\,, \tag{C.12}$$

where the sum is over the $70 = 8!/4!4!$ distributions of 8 elements into two groups of 4. Integrating over $\hat\theta$ and $\check\theta$ gives

$$\sum_{(70)} t^{i_1r_1\cdots i_4r_4}t^{i_5r_5\ldots i_8r_8}L_{i_1r_1}\cdots L_{i_8r_8} = \\ = \sum_{(70)}((6)\delta^{1234} - (3)\delta^{12}\delta^{34})((6)\delta^{5678} - (3)\delta^{56}\delta^{78})L_{(1)}\cdots L_{(4)}\,. \tag{C.13}$$

Performing the contractions gives the expression

$$\mathcal{A}^4\, W_B(X) = [6L4 - 4L22]\,. \tag{C.14}$$

Similar manipulations are needed to determine the contributions from terms with one, two, three or four factors of ε in the expansion of the $\mathcal{A}^4$ term. These are to be added together to get the total contribution. The result is that these terms give a contribution to the linear divergence of the form

$$\mathcal{D}3 = \frac{1}{4!}\int_0^\infty d|X_T|\,|X_T|^2\,\mathcal{A}^4\,W_B(X) = \frac{80}{27}\,, \tag{C.15}$$

while the finite contribution from these terms is

$$\mathcal{F}3 = \frac{1}{4!}\int_0^\infty d|X_T|\,|X_T|^2\,\sqrt{X_T^2 + R^2}\,\mathcal{A}^4\,W_B(X) = -\frac{176}{81}\,. \tag{C.16}$$

The total coefficient of the linear divergence arising from the sum of all terms is

$$\mathcal{D} = \mathcal{D}1 + \mathcal{D}2 + \mathcal{D}3 = \frac{21127}{8640}\,, \tag{C.17}$$

and the total finite result is

$$\mathcal{F} = \mathcal{F}1 + \mathcal{F}2 + \mathcal{F}3 = \frac{2}{405}\,. \tag{C.18}$$

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